INTEGRATION IN LOCALLY COMPACT HAUSDORFF SPACES

A THESIS

Presented to

The Faculty of the Graduate Division

by

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

June, 1966

INTEGRATION IN LOCALLY COMPACT HAUSDORFF SPACES

Approved:

Chairman

Date approved by Chairman: May 23, 1966

ACKNOWLEDGMENTS

The author would like to express thanks to his thesis advisor, Dr. Eric R. Immel, whose guidance, inspiration, and patience made this undertaking possible.

The author also wishes to thank the other members of the committee, Doctors Robert H. Kasriel and Joseph Ford of the School of Physics, for their kindness and help.

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CHAPTER I

INTRODUCTION

The purpose of this study is to consider integration of real-valued functions on locally compact Hausdorff spaces. There are two basic approaches to the theory of integration of functions defined on topological spaces; one considers an arbitrary topological space and assumes that there is given a *measure* on a certain σ-algebra of subsets of the space. By *measure* we understand a set function μ with values in the positive extended real number system which has the properties that $\mu(A) \leq \mu(B)$ if $A \subseteq B$ and $\mu(\bigcup_{n\geq 1} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ where (A_n) is a sequence of pairwise disjoint sets in the given σ-algebra of subsets.

The other approach assumes that the domain space is a locally compact Hausdorff space and that there is given a linear functional on the vector space of continuous real or complex-valued functions each of which vanishes outside some compact set. The functional is then extended by some procedure to a larger class of functions defined on the space.

On the surface, the first approach appears to be far more general than the second. However, Kakutani has shown that in a certain sense integration on locally compact Hausdorff spaces is equivalent to integration on abstract spaces. We shall consider the second approach in this study.

The motivation for considering a linear functional on a locally compact Hausdorff space is connected with a classical theorem of F. Riesz.

This theorem states that if μ is a positive linear functional on the space $C_r^o(R)$ of real-valued continuous functions, each of which vanishes outside a compact interval of the real line R (the interval depending on the function), then there exists a real-valued increasing function α such that

$$\mu(f) = \int_{-\infty}^{\infty} f \, d\alpha,$$

for all f in $C_r^o(R)$.

The original theorem of Riesz was stated for bounded linear functionals on the space of continuous functions vanishing outside an interval; the above theorem is a corrollary [5].

In this study, we shall follow the ideas of Bourbaki as developed in their monograph on integration [1]. The second chapter discusses some of the necessary material from the theory of semicontinuous functions. The third chapter develops the extension of a linear functional and establishes some of the properties of the extended functional obtained. The fourth chapter then shows how to obtain a theory of measure on a class of subsets of the domain space and relates the theory given in this study to the other approach to integration mentioned above. It concludes with a generalization, in a form due to Bourbaki, of the Lebesgue-Radon-Nikodym theorem. This generalization is a generalization to arbitrary locally compact Hausdorff spaces which are not σ-compact, i.e., cannot be written as the union of a sequence of compact sets. It is the aim throughout to give a reasonably self-contained account of the subject, and to supply very detailed proofs, some of which appear to be difficult to locate in such form in the literature. A reference found especially helpful was Nachbin [6].

As examples of linear functionals to which the extension theory applies, consider first any space with the discrete topology. All real-valued functions defined on such a space are continuous, and consequently the continuous functions vanishing outside of a compact set are those that vanish outside a finite set. If f is such a function, we define $\mu(f) = \sum_{x \in T} f(x)$, where $T = \{x: f(x) \neq 0\}$. It is obvious that μ is a positive linear functional, i.e., $\mu(f) \geq 0$ whenever $f \geq 0$. The extension procedure can now be applied yielding the space of functions which generate, in a certain sense, absolutely convergent series. The integral obtained will then be the sum of the series.

Another example is the ordinary Riemann integral on the space of continuous functions each of which vanishes outside a compact set in the real line. An easy consequence of results to be developed later shows that the extension of this linear functional yields the ordinary Lebesgue integral on the real line. The integrable functions will be real-valued Lebesgue integrable functions. The proof of this fact follows at once from the fact that in both theories, for each integrable function f there exists a sequence of continuous functions (f_n), each vanishing outside a compact set, such that

$$\int |f - f_n| dx \to 0 \ , \quad n \to \infty \ ,$$

and the fact that real-valued functions equal "almost everywhere" to integrable functions are integrable.

A similar example is the extension of a linear functional μ given by $\mu(f) = \int f d\alpha$ where f vanishes outside a compact set, and α is an increasing function. Here the integral used is the Riemann-Stieltjes integral. The extension yields the Lebesgue-Stieltjes integral in the real line.

CHAPTER II

SEMICONTINUOUS FUNCTIONS

The purpose of this chapter is to introduce the concept of semicontinuity and to establish some of the elementary properties of semicontinuous functions which will be needed later in the development of the integral.

In the following discussion R will denote the set of all real numbers, and R^+ will denote the set of all non-negative real numbers. In addition we shall also use R^* to denote the *extended real number system* made out of the real number system R compactified in the usual way by adding two points, ∞ and $-\infty$. The usual conventions regarding the topology of R^* and the operations of addition and multiplication in R^* will be observed. We shall also set $R^{*+} = R^{*}\cup\{\infty\}$.

If X is an arbitrary topological space, we shall use $C_r(X)$ to denote the set of all real-valued continuous functions with domain X. In addition, if f is a function whose domain is X and whose range is a subset of R^*, then f will be called an *extended-real-valued* function. The set of all extended-real-valued continuous functions defined on X will be denoted by $C_{r^*}(X)$.

We shall use $C_r^+(X)$ to denote the set of all non-negative real-valued functions defined on X and $C_{r^*}^+(X)$ to denote all non-negative extended-real-valued continuous functions defined on X.

Definition 2.1. Let f be an extended-real-valued function defined on a topological space X, and let $x_o \in X$. Then f is said to be *lower semicontinuous at the point* x_o if and only if (1) $f(x_o) = -\infty$ or (2) if $f(x_o) > -\infty$, then for every $\varepsilon > 0$ there exists a neighborhood V of x_o such that $f(x_o) - \varepsilon < f(x)$ for all x in V.

The usual definition of continuity for real-valued functions specifies that for every $\varepsilon > 0$ there exists a neighborhood V of x_o such that $f(x_o) - \varepsilon < f(x) < f(x_o) + \varepsilon$ for all x in V. It can be seen that lower semicontinuity for real-valued functions is merely a weakening of continuity which has the effect of forcing f(x) to stay close to $f(x_o)$ if f(x) is below $f(x_o)$ but not if f(x) is above $f(x_o)$.

Definition 2.2. Let f be an extended-real-valued function defined on a space X. Then f is said to be *lower semicontinuous on X* if and only if for each $x_o \in X$, f is lower semicontinuous at x_o.

It is obvious that the definition of lower semicontinuity at a point x_o is equivalent to the statement that either $f(x_o) = -\infty$ or, if $f(x_o) > -\infty$, then for every real number $h < f(x_o)$ there exists a neighborhood V of x_o such that $h < f(x)$ for all $x \in V$.

Upper semicontinuity may be defined in a similar way; however, this concept will not be used in the development that follows.

We start the development of lower semicontinuous functions by giving a characterization of them in terms of certain subsets of the space X. Then we develop the properties of families and sequences of lower semicontinuous functions.

<u>Theorem 2.3.</u> Let f be an extended-real-valued function defined on a topological space X. Then f is lower semicontinuous on X if and only if, for every real number h, the set $\{x \in X: h < f(x)\}$ is open.

<u>Proof.</u> Suppose f is a lower semicontinuous function on X. Let h be a real number. If the set $\{x \in X: h < f(x)\}$ is empty, then it is open. Otherwise, suppose $x_o \in \{x \in X: h < f(x)\}$. Then $h < f(x_o)$. By definition of lower semicontinuity at x_o, either $f(x_o) = -\infty$ or there exists a neighborhood V of x_o such that $h < f(x)$ for all x in V. Here $f(x_o) \neq -\infty$, for otherwise it would be false that $h < f(x_o)$. Hence $V \subseteq \{x \in X: h < f(x)\}$. Thus this set contains a neighborhood of every one of its points. Hence it is an open set.

Conversely, suppose that the set $\{x \in X: h < f(x)\}$ is open for all real numbers h. Let $x_o \in X$. If $f(x_o) = -\infty$, then f is lower semicontinuous at x_o. If $f(x_o) \neq -\infty$, then for some real number h, $x_o \in \{x \in X: h < f(x_o)\}$. This set is open by assumption. Hence there exists a neighborhood V of x_o such that $V \subseteq \{x \in X: h < f(x_o)\}$. In other words, $h < f(x)$ for all x in V. Thus f is lower semicontinuous at x_o. It follows that f is lower semicontinuous on X. $\square$

<u>Theorem 2.4.</u> If f_i (for $i = 1, \ldots, n$) is an extended-real-valued lower semicontinuous function defined on X, then the function f defined by

$$f(x) = \inf \{f_1(x), \ldots, f_n(x)\}$$

is lower semicontinuous on X.

Proof. Let $x_o \varepsilon X$. If $f(x_o) = -\infty$, then f is lower semicontinuous at x_o. Hence, suppose that $h < f(x_o)$ for some real number h. Then by definition of $f(x_o)$, for every index i, $h < f_i(x_o)$. Since each f_i is lower semicontinuous at x_o, then for each index i there exists a neighborhood V_i such that $h < f_i(x)$ for all x in V_i. Let $V = \bigcap_{i=1}^{n} V_i$. Then V is a neighborhood of x_o, and for all x in V, and all i = 1, 2, ..., n, $h < f_i(x)$. Hence

$$h < \inf \{f_i(x), \ldots, f_n(x)\} = f(x)$$

for all x in V. Thus f is lower semicontinuous at x_o. Since x_o was chosen arbitrarily in X, it follows that f is lower semicontinuous on X. ▯

Theorem 2.5. Let $\mathcal{F}$ be a family of lower semicontinuous functions defined on X. Let F be defined on X by

$$F(x) = \sup \{f(x): f \varepsilon \mathcal{F}\}$$

for all $x \varepsilon X$. Then f is lower semicontinuous on X.

Proof. Let $x_o \varepsilon X$. It suffices to show that if $F(x_o) \nleq -\infty$, then F is lower semicontinuous at x_o. Let h be a real number such that $h < F(x_o)$. Then there exists an $f \varepsilon \mathcal{F}$ such that $h < f(x_o)$. By lower semicontinuity of f at x_o, there exists a neighborhood V of x_o such that $h < f(x)$ for all $x \varepsilon V$. Since $f(x) \leq F(x)$ for all $x \varepsilon X$, $h < F(x)$ for all $x \varepsilon V$. Hence F is lower semicontinuous at x_o. It follows that F is lower semicontinuous on X. ∎

Theorem 2.6. Let f and g be two lower semicontinuous functions defined on X. Let x_o be a point at which f + g is defined. Then f + g is lower semicontinuous at x_o.

Proof. It suffices to suppose that $f(x_o) + g(x_o) \neq -\infty$. Let h be a real number such that $h < f(x_o) + g(x_o)$. Then $h - g(x_o) < f(x_o)$. Choose h' such that $h - g(x_o) < h' < f(x_o)$. Define $h'' = h - h'$. Now $h - g(x_o) < h'$; $-g(x_o) < h' - h$, and $h'' = h - h' < g(x_o)$. Since f and g are both lower semicontinuous at x_o, we choose V' and V" to be neighborhoods of x_o such that $h' < f(x)$ for all $x \in V$ and $h'' < g(x)$ for all x in V". Let $V = V' \cap V''$. Then

$$h = h'' + h' < f(x) + g(x)$$

for all x in V. Thus f + g is lower semicontinuous at x_o. ∎

Theorem 2.7. Let (f_i) be a sequence of non-negative lower semicontinuous functions on X. Then the function f defined on X by

$$f(x) = \sum_{i=1}^{\infty} f_i(x), \quad x \in X$$

is lower semicontinuous on X.

Proof. Note that f(x) always exists in the extended-real sense. Let

$$\bar{f}_j(x) = \sum_{i=1}^{j} f_i(x).$$

Then $\bar{f}_j$ will be lower semicontinuous for each j by Theorem 2.5. Note that $f(x) = \sup_j \{\bar{f}_j(x)\}$, for each $x \in X$. Hence f is lower semicontinuous on X by Theorem 2.4. ▯

<u>Theorem 2.8</u>. Let f be an extended-real-valued function defined on X. Let $\lambda > 0$ be a real number. Then λf is lower semicontinuous on X.

This result is an immediate consequence of the definition of lower semicontinuity. We have now developed all of the general properties of lower semicontinuous functions that will be needed later. We close this chapter with two results which are not true in general but which are true if the space X enjoys certain special properties.

A Hausdorff space X is said to be *completely regular* [3] if for each $x \in X$ and each neighborhood V of x there exists a real-valued continuous function f mapping X into the closed unit interval such that $f(x) = 1$ and $f(X - V) = \{0\}$ It can be shown that if X is a locally compact Hausdorff space, A a compact subset of X, and U a neighborhood of A, then there exists a real-valued continuous function f mapping X into the closed unit interval such that $f(A) = \{1\}$ and $f(X - U) = \{0\}$*. In particular, this implies that a locally compact Hausdorff space is completely regular.

<u>Theorem 2.9</u>. Let f be an extended-real-valued function defined on a completely regular space X. Then

$$f(x) = \sup \{g(x) : g \in C_{r^*}(X),\ g \leq f\}$$

*Reference 3, page 146.

if and only if f is lower semicontinuous on X.

Proof. Let

$$F(x) = \sup \{g(x): g \in C_{r*}(X), g \leq f\}.$$

Suppose that $f = F$. Then f is lower semicontinuous on X by Theorem 2.5.

Assume now that f is lower semicontinuous on X. By definition $F \leq f$. Suppose that for some x_o in X, $f(x_o) = -\infty$. Then since $F(x_o) \leq f(x_o) = -\infty$, we have that $F(x_o) = -\infty$. Thus, for those points $x \in X$ such that $f(x) = -\infty$, we have that $F(x) = f(x)$. We now consider those points $x \in X$ such that $f(x) \neq -\infty$. Let x_o be such a point. Let h be any real number such that $h < f(x_o)$. There exists a neighborhood V of x_o such that $h < f(x)$ for all x in V. Since X is completely regular, there exists a function $g \in C_r(x)$ such that $0 \leq g(x) \leq 1$ for all $x \in X$, $g(x_o) = 1$ and $g(X - V) = \{0\}$. Consider the function ϕ defined on [0,1] by

$$\phi(x) = \begin{cases} (h + 1) - 1/x, & x \neq 0 \\ -\infty, & x = 0. \end{cases}$$

Then ϕ maps [0,1] onto $[-\infty, h]$, $\phi(0) = -\infty$, $\phi(1) = h$, and $\phi \in C_{r*}([0,1])$. Define the function G on X by $G(x) = \phi(g(x))$ for all $x \in X$. Since G is the composition of two functions in $C_{r*}(X)$ and $C_{r*}([0,1])$, respectively, $G \in C_{r*}(X)$. Suppose $x \in V$. Then $G(x) = \phi(g(x)) \leq h < f(x)$ for all x in V. If $x \in X - V$, then $G(x) = \phi(g(x)) = \phi(1) = h$. Hence, for all $x \in X$, $G(x) \leq h < f(x)$. By definition of F, $G(x) \leq F(x)$; in particular

$$F(x_o) \geq G(x_o) = \phi(g(x_o)) = \phi(1) = h.$$

Hence, $F(x_o) \geq h$ for all $h < f(x_o)$. Thus $F(x_o) \geq f(x_o)$. Since $F(x_o) \leq f(x_o)$, it follows that $F(x_o) = f(x_o)$. Thus $F = f$. $\square$

If X is a locally compact Hausdorff space, a stronger result may be obtained for non-negative extended-real-valued functions. In this case, the functions g which were majorized by f in the previous theorem may be taken to be real-valued continuous functions vanishing outside a compact set. We shall show this fact, but first a preliminary definition is needed.

Definition 2.10. Let f be an extended-real-valued function defined on a topological space X. The closure of the set $\{x \in X\colon f(x) \neq 0\}$ is called the support of f.

We shall use the sumbol $C_r^o(X)$ to represent the set of all real-valued continuous functions defined on X having compact supports in X. The symbol $C_r^{o+}(X)$ will denote all the non-negative functions in $C_r^o(X)$.

Theorem 2.11. Let f be a non-negative extended-real-valued function defined on a locally compact Hausdorff space X. Then

$$f(x) = \sup \{\phi(x)\colon \phi \in C_r^{o+}(X),\ \phi \leq f\}$$

for all $x \in X$ if and only if f is lower semicontinuous on X.

Proof. Let

$$F(x) = \sup \{\phi(x) : \phi \in C_r^{o+}(X),\ \phi \le f\}$$

for all $x \in X$. Suppose $f(x) = F(x)$. Then f is lower semicontinuous on X by Theorem 2.5.

Now, suppose that f is lower semicontinuous on X. Clearly $f(x) \ge F(x)$ for all $x \in X$. Consider a fixed point $x_o \in X$. If $f(x_o) = 0$, then $F(x_o) = f(x_o)$. If $f(x_o) > 0$, we may choose a real number h such that $0 < h < f(x_o)$. Since f is lower semicontinuous there is a neighborhood V of x_o such that $h < f(t)$ for all $t \in V$. In a locally compact Hausdorff space, the compact neighborhoods of a point constitute a basis for the neighborhood system of that point [3]. Hence, there exists a compact neighborhood U of x_o such that $U \subsetneq V$. There exists a real-valued continuous function g from X into $[0,1]$ such that $g(x_o) = 1$ and $g(X - U) = \{0\}$, recalling the remark that X in this case is completely regular. Define $\psi(x) = h \cdot g(x)$ for all x in X. If $x \in U$, then $\psi(x) = h \cdot g(x) \le h < f(x)$. If $x \notin U$, then $\psi(x) = 0$. Hence, for all $x \in X$, $\psi(x) \le f(x)$. The support of ψ is contained in U, and hence $\psi \in C_r^{o+}(X)$. Finally, $\psi(x_o) = h \cdot g(x_o) \le F(x_o)$ by definition of F. Thus $h \le F(x_o) \le f(x_o)$ for all $h < f(x_o)$. Hence $F(x_o) = f(x_o)$. Since x_o is arbitrary in X, $F(x) = f(x)$ for all $x \in X$. $\square$

CHAPTER III

THEORY OF INTEGRATION

Throughout this chapter and succeeding ones, X will denote a locally compact Hausdorff space. We shall first show that $C_r^O(X)$ is a vector space, and develop the elementary properties of positive linear functionals on such spaces. We then assume that there is a given positive* linear functional on $C_r^O(X)$, and we show how to extend the domain of this functional to a larger space that has several properties which $C_r^O(X)$ lacks. In all that follows we shall use the symbol $LS_r(X)$ to denote the set of all real-valued lower semicontinuous functions on X and $LS_{r*}(X)$ to denote the set of all extended-real-valued lower semicontinuous functions on X.

Theorem 3.1. The collection of functions $C_r^O(X)$ is a linear manifold over R.

Proof. Let f and g be in $C_r^O(X)$. It is a well known fact that f + g is continuous. Let A denote the support of f and B the support of g. Now

$$\{x \in X\colon f(x) + g(x) \neq 0\} \subseteq \{x \in X\colon f(x) \neq 0\} \cap \{x \in X\colon g(x) \neq 0\}.$$

For any two sets C and D, $\overline{C \cup D} = \bar{C} \cup \bar{D}$ and, if $C \subseteq D$, then $\bar{C} \subseteq \bar{D}$. Hence the support of f + g is a closed subset of $A \cup B$. Since $A \cup B$ is compact,

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*Here and in what follows, the term *positive* is used in the sense of *non-negative*, unless otherwise stated.

the support of $f + g$ is also compact. Hence $f + g \in C_r^o(X)$. Clearly, if λ is a real number and $f \in C_r^o(X)$, then $\lambda \cdot f \in C_r^o(X)$. In particular, $(-1)f \in C_r^o(X)$. Since $C_r^o(X)$ is a subset of the space of continuous functions on X, it is a linear manifold in that space. ▯

From the proof of the preceding theorem, it is obvious that if f and g $\in C_r^o(X)$ and if λ is a positive real number, then $f + g$ and $\lambda \cdot f \in C_r^{o+}(X)$.

Theorem 3.2. Suppose $f \in C_r^o(X)$. Then $|f| \in C_r^{o+}(X)$. Furthermore, the functions

$$f_1 = \max \{f, 0\}, \quad f_2 = -\min \{f, 0\}$$

are in $C_r^{o+}(X)$.

Proof. The support of $|f|$ is the same as that of f, and continuity of $|f|$ follows at once by continuity of the absolute value function and preservation of continuity under composition. Hence $|f| \in C_r^{o+}(X)$. Since $C_r^o(X)$ is a linear manifold, the relations

$$\max \{f, 0\} = \frac{|f| + f}{2} \geq 0,$$

and

$$-\min \{f, 0\} = \frac{|f| - f}{2} \geq 0,$$

show that f_1 and f_2 are in $C_r^{o+}(X)$. ▯

Definition 3.3. A real-valued linear functional μ on $C_r^o(X)$ which is positive on $C_r^{o+}(X)$ will be called an integral on $C_r^o(X)$.

Theorem 3.4. Suppose that f and g ε $C_r^o(X)$ and $f \leq g$. Then $\mu(f) \leq \mu(g)$.

Proof. Note that $g - f \geq 0$. Hence $g - f \; \varepsilon \; C_r^{o+}(X)$. Thus $\mu(g - f) \geq 0$, which implies that $\mu(f) \leq \mu(g)$. ▯

Theorem 3.5. Let μ be a positive real-valued functional which is additive and positively homogeneous on $C_r^{o+}(X)$. Then there exists one and only one extension $\bar{\mu}$ of μ to $C_r^o(X)$ which is an integral on $C_r^o(X)$.

Proof. Suppose that f ε $C_r^o(X)$. Define f_1 and f_2 as in Theorem 3.2. Then $f = f_1 - f_2$. Define $\bar{\mu}$ on $C_r^o(X)$ by $\bar{\mu}(f) = \mu(f_1) - \mu(f_2)$ for all f ε $C_r^o(X)$.

Now let f ε $C_r^o(X)$ and suppose that f_3 and f_4 are in $C_r^{o+}(X)$ such that $f = f_3 - f_4$. Then $f_1 + f_4 = f_2 + f_3$, and since μ is additive, $\mu(f_1) + \mu(f_4) = \mu(f_2) + \mu(f_3)$. It thus follows that

$$\bar{\mu}(f) = \mu(f_1) - \mu(f_2) = \mu(f_3) - \mu(f_4).$$

Now let f and g ε $C_r^o(X)$. Then

$$f + g = (f_1 - f_2) + (g_1 - g_2) = (f_1 + g_1) - (f_2 + g_2),$$

and $f_1 + g_2 \; \varepsilon \; C_r^{o+}(X)$, and $f_2 + g_2 \; \varepsilon \; C_r^{o+}(X)$. Using the preceding remarks, we obtain

$$\bar{\mu}(f + g) = \mu(f_1 + g_1) - \mu(f_2 + g_2) = \bar{\mu}(f) + \bar{\mu}(g).$$

Hence $\bar{\mu}$ is additive on $C_r^o(X)$.

Suppose λ is a non-negative real number. Then, for all $f \varepsilon C_r^o(X)$,

$$\bar{\mu}(\lambda \cdot f) = \bar{\mu}(\lambda(f_1 - f_2)) = \bar{\mu}(\lambda \cdot f_1 - \lambda f_2) = \mu(\lambda \cdot f_1) - \mu(\lambda \cdot f_2)$$

$$= \lambda \cdot \mu(f_1) - \lambda \cdot \mu(f_2) = \lambda[\mu(f_1) - \mu(f_2)] = \lambda \cdot \bar{\mu}(f).$$

If $\lambda < 0$, then

$$\bar{\mu}(\lambda f) = \bar{\mu}(-|\lambda| f) = |\lambda| \cdot \bar{\mu}(-f) = |\lambda| \cdot \bar{\mu}(f_2 - f_1).$$

Note that $-f \varepsilon C_r^o(X)$ and $f_2 - f_1$ is the representation of $(-f)$ as in Theorem 3.2. Hence

$$|\lambda| \cdot \bar{\mu}(f_2 - f_1) = |\lambda| \cdot [\mu(f_2) - \mu(f_1)] = -|\lambda| \cdot [\mu(f_1) - \mu(f_2)]$$

$$= \lambda[\mu(f_1) - \mu(f_2)] = \lambda \cdot \bar{\mu}(f).$$

It follows that $\bar{\mu}$ is a linear functional on $C_r^o(X)$ and that $\bar{\mu}$ agrees with μ on $C_r^{o+}(X)$. If $\tilde{\mu}$ is any integral on $C_r^o(X)$ which agrees with μ on $C_r^{o+}(X)$, then necessarily

$$\tilde{\mu}(f) = \tilde{\mu}(f_1 - f_2) = \tilde{\mu}(f_1) - \tilde{\mu}(f_2) = \mu(f_1) - \mu(f_2) = \mu(f).$$

Hence $\bar{\mu}$ is the unique extension of μ to $C_r^o(X)$. ▯

If K is a compact subset of X, the collection of all functions in $C_r^o(X)$ whose support is contained in K will be denoted by $C_r(X; K)$. The collection of all functions in $C_r(X; K)$ which are also members of $C_r^{o+}(X)$ will be denoted by $C_r^+(X; K)$.

Theorem 3.6. The set $C_r(X; K)$, where K is a compact subset of X, is a linear manifold in $C_r^o(X)$.

This result follows at once from the definitions.

It is easy to see that if f and g are in $C_r^+(X; K)$ and λ is a non-negative number, then λf and $f + g$ are in $C_r^+(X; K)$. Actually, the vector space property of $C_r(X; K)$ does not depend upon the fact that K is compact. However, the next result to be obtained requires that K be compact, and we shall never consider the more general case.

Definition 3.7. Let $f \in C_r^o(X)$. Then we define

$$\|f\|_\infty = \sup \{|f(x)|: x \in X\}.$$

The function $\|\cdot\|_\infty$ is called the uniform or sup norm. The following theorem asserts that the linear manifold $C_r(X; K)$, where K is a compact subset of X, is a vector subspace of $C_r^o(X)$ with respect to the topology induced on X by the uniform norm. We use the term *subspace* in the restricted sense; that is, we apply the term only to *closed linear manifolds*.

Theorem 3.8. Let K be a compact subset of X. Then given a sequence (f_n) of functions in $C_r(X; K)$ such that for every $\varepsilon > 0$ there exists a positive integer N such that for all m and $n \geq N$, $\|f_m - f_n\|_\infty < \varepsilon$, then the pointwise limit exists and is in $C_r(X; K)$.

Proof. Let $x \in X$. It follows directly from the hypotheses that $(f_n(x))$ is a Cauchy sequence of real numbers, and therefore convergent. Hence the pointwise limit f exists. If $x \notin K$, then $\lim_{n\to\infty} f_n(x) = 0$. It follows that the support of f is contained in K and thus is compact. Note that the hypotheses describe uniform convergence of the sequence (f_n). It follows that f is continuous on X. Thus $f \in C_r(X; K)$. $\square$

Theorem 3.9. Let μ be an integral on $C_r^o(X)$, and let K be a compact subset of X. Then the restriction of μ to $C_r(X; K)$ is a continuous functional relative to the uniform norm.

Proof. Since X is a locally compact Hausdorff space, there exists a compact neighborhood V of K. Consequently, there exists a real-valued continuous function F mapping X into $[0, 1]$ such that $F(K) = \{1\}$ and $F(X - V) = \{0\}$. Suppose $f \in C_r(X; K)$. Then

$$- \|f\|_\infty F(x) \leq f(x) \leq \|f\|_\infty F(x)$$

for all $x \in X$. Hence, since $F \in C_r^o(X)$,

$$- \|f\|_\infty \mu(F) \leq \mu(f) \leq \|f\|_\infty \mu(f);$$

that is,

$$|\mu(f)| \leq \|f\|_\infty \mu(F).$$

Now $\mu(F) < \infty$. Hence, for any sequence (f_n) with $\|f_n\| \to 0$ we have $\lim_{n\to\infty} \mu(f_n) = 0$. It follows that μ is continuous at the zero function. Since μ is a linear functional, this fact implies continuity on the entire space $C_r(X; K)$. ▯

We shall now show how to extend μ to a much larger space containing $C_r^o(X)$.

Definition 3.10. Let $f \in LS_{r*}^+(X)$. We define

$$\mu^*(f) = \sup \{\mu(\phi) : \phi \in C_r^{o+}(X), \quad \phi \le f\}.$$

Theorem 3.11. Let $f \in LS_{r*}^+(X)$. Then

(1) $0 \le \mu^*(f) \le \infty$,

(2) $\mu(f) = \mu^*(f)$ if $f \in C_r^{o+}(X)$,

(3) $\mu^*(f) \ge \mu^*(g)$ for all f and $g \in LS_{r*}^+(X)$ such that $f \ge g$,

(4) If $\lambda \ge 0$, then $\mu^*(\lambda f) = \lambda \cdot \mu^*(f)$.

Proof. (1) Since μ is positive on $C_r^{o+}(X)$, it follows at once that $\mu^*(f) \ge 0$.

(2) This part follows at once from the definition of $\mu^*(f)$.

(3) Suppose that f and $g \in LS_{r*}^+(X)$ and that $f \ge g$. Then, for all functions $\phi \in C_r^{o+}(X)$ such that $\phi \le g$, we have that $\phi \le f$. Hence

$$\{\mu(\phi) : \phi \in C_r^{o+}(X), \quad \phi \le g\} \subseteq \{\mu(\phi) : \phi \in C_r^{o+}(X), \ \phi \le f\}.$$

It follows that $\mu^*(g) \le \mu^*(f)$.

(4) If $\lambda > 0$ and if $f \in LS^+_{r*}(X)$, then $\lambda f \in LS^+_{r*}(X)$;

$$\mu^*(\lambda f) = \sup \{\mu(\phi) : \phi \in C_r^{o+}(X),\ \ \phi \le f\} =$$

$$\sup \{\mu(\lambda\phi) : \phi \in C_r^{o+}(X),\ \ \lambda\phi \le \lambda f\} = \sup \{\lambda\mu(\phi) : \phi \in C_r^{o+}(X),\ \phi \le f\}$$

$$= \lambda \sup \{\mu(\phi) : \phi \in C_r^{o+}(X),\ \ \phi \le f\} = \lambda\mu^*(f).$$

If $\lambda = 0$, the conclusion is obvious. ▯

Definition 3.12. Let F be a family of extended-real-valued functions defined on X. If, given any two functions f_1 and f_2 in F, there exists a third formula $f \in F$ such that $f_1 \le f$ and $f_2 \le f$, then F is said to be *directed upward.*

The concept of a family of functions *directed downward* is defined analogously.

Theorem 3.13.* Let G be an upward directed subset of $LS^+_{r*}(X)$. If $f(x) = \sup \{g(x) : g \in G\}$ for all $x \in X$, then $f \in LS^+_{r*}(X)$ and $\mu^*(f) = \sup \{\mu^*(g) : g \in G\}$.

Proof. It follows from Theorem 2.5 that $f \in LS^+_{r*}(X)$. Now if $g \in G$, then $g \le f$ and hence

$$\mu^*(f) \ge \mu^*(g).$$

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*From now on we shall assume a given positive linear functional μ on $C_r^o(X)$.

It follows that

$$(1) \qquad \mu^*(f) \geq \sup \{\mu^*(g) : g \in G\}.$$

We shall now establish the opposite inequality. The proof is rather lengthy. It will first be assumed that $f \in C_r^{o+}(X)$ and $G \subseteq C_r^{o+}(X)$; the general case will then be reduced to this case. Let $\varepsilon > 0$ be given, and fix $x \in X$. By definition of f there exists a function $g_x \in G$ such that $g_x(x) > f(x) - \varepsilon/3$. Now f is continuous at x. Hence there exists an open neighborhood V_x' of x such that $f(t) < f(x) + \varepsilon/3$ for all $t \in V_x'$. Thus

$$g_x(x) > f(x) - \varepsilon/3 > f(t) - 2\varepsilon/3$$

for all $t \in V_x'$. Since g_x is also continuous at x, there exists an open neighborhood V_x'' such that $g_x(x) - \varepsilon/3 < g_x(t)$ for all $t \in V_x''$. Hence, if $V_x = V_x' \cap V_x''$, then V_x is an open neighborhood of x, and $g_x(t) > f(t) - \varepsilon$ for all $t \in V_x$. Now apply the above argument to each x in K, the support of f. Each neighborhood is open, and consequently the collection of all such neighborhoods is an open covering of K. Since K is compact, there is a finite number of neighborhoods $V_{x_1}, \ldots, V_{x_n}$ which cover K. Since G is directed upward, there is a function $g \in G$ such that $g \geq g_{x_i}$, $i = 1, 2, \ldots, n$. Consider a point $z \in K$. Then $z \in V_{x_i}$ for some i, and $g(z) \geq g_{x_i}(z) > f(z) - \varepsilon$, that is, $g(z) > f(z) - \varepsilon$ for all $z \in K$. If $z \in X - K$, then $f(z) = 0$ and hence $g(z) = 0$. It follows that $g(z) > f(z) - \varepsilon$ for all $z \in X$. Hence, noting that $f(z) \geq g(z)$ for all

$z \in X$,

$$\|f - g\|_\infty = \sup \{|f(x) - g(x)|: x \in X\} \leq \varepsilon.$$

Furthermore, since $0 \leq f \leq g$, the support of g is contained in K. Thus, corresponding to each positive integer $n \geq 1$, there exists a function $g_n \in G$ with support contained in K such that $\|f - g_n\|_\infty < 1/n$, that is, $g_n \to f$ uniformly on X. Applying Theorem 3.9, we obtain $\mu(g_n) \to \mu(f)$. But $f \in C_r^{o+}(X)$. Hence $\mu^*(f) = \mu(f)$, and $\mu^*(g_n) = \mu(g_n)$ for each $n \geq 1$. Hence $\lim_{n\to\infty} \mu^*(g_n) = \mu^*(f)$. But $\lim_{n\to\infty} \mu^*(g_n) \leq \sup \{\mu^*(g): g \in G\}$. It follows that $\mu^*(f) \leq \sup \{\mu^*(g): g \in G\}$. Noting (1) above, we obtain that $\mu^*(f) = \sup \{\mu^*(g): g \in G\}$.

Now suppose that no extra restrictions have been placed on f and G. We shall show that for every ψ in $C_r^{o+}(X)$ such that $\psi \leq f$, we have $\mu(\psi) \leq \sup \{\mu^*(g); g \in G\}$. Then from the definition of $\mu^*(f)$,

$$\mu^*(f) = \sup \{\mu(\psi): \psi \in C_r^{o+}(X),\ \psi \leq f\} \leq \sup \{\mu^*(g): g \in G\},$$

from which the desired equality follows immediately. Define $\Phi_g = \{\phi \in C_r^{o+}(X): \phi \leq g\}$, $g \in G$ and let $\Phi = \bigcup_{g \in G} \Phi_g$. Now $\Phi \subseteq C_r^{o+}(X)$. We shall show that $f = \sup \{\phi: \phi \in \Phi\}$. Suppose $\phi \in \Phi$. Then there is a function $g \in G$ such that $\phi \leq g \leq f$. Hence

$$\text{(2)} \qquad f \geq \sup \{\phi: \phi \in \Phi\}.$$

Fix $x \in X$. Let $\varepsilon > 0$ be given. Noting the definition of f, we obtain a function $g \in G$ such that $f(x) - \varepsilon < g(x)$. But by Theorem 2.11, $g = \sup \{\phi: \phi \in \Phi_g\}$, since $g \in LS^+_{r*}(X)$. Hence

$$f(x) - \varepsilon < \sup \{\phi(x): \phi \in \Phi_g\}.$$

Thus, for this particular x, we have by (2) that

$$f(x) - \varepsilon \leq \sup \{\phi(x): \phi \in \Phi_g\} \leq \sup \{\phi(x): \phi \in \Phi\} \leq f(x)$$

for all $\varepsilon > 0$. Thus $f(x) = \sup \{\phi(x): \phi \in \Phi\}$. Since x was chosen arbitrarily in X, we have

$$f = \sup \{\phi: \phi \in \Phi\}. \tag{3}$$

Let ψ be any function in $C^{o+}_r(X)$ such that $\psi \leq f$. For each $\phi \in \Phi$, define $\lambda = \min\{\psi, \phi\}$. Then $\lambda \in C^{o+}_r(X)$. It will be shown that the collection of all such λ's forms an upward directed subset of $C^{o+}_r(X)$ and that $\psi = \sup \{\lambda: \lambda = \min \{\phi, \psi\}, \phi \in \Phi\}$, from which we will infer that $\mu(\psi) \leq \sup \{\mu^*(g): g \in G\}$. Let Λ denote the collection of all such λ's. For each $\lambda \in \Lambda$, $\psi \geq \lambda = \min \{\psi, \phi\}$. Hence $\psi \geq \sup \{\lambda: \lambda \in \Lambda\}$. Fix $x \in X$. If $\psi(x) < f(x)$, then by (3) there exists a $\phi \in \Phi$ such that $\psi(x) < \phi(x) \leq f(x)$. Hence, for this x, $\psi(x) = \min \{\phi(x), \psi(x)\} = \lambda(x)$ and thus, $\psi(x) \leq \sup \{\lambda(x): \lambda \in \Lambda\}$. But, if $\psi(x) = f(x)$, then $\psi(x) = \sup \{\phi(x): \phi \in \Phi\}$, and, for this x,

$$\{\phi(x): \phi \in \Phi\} = \{\lambda(x): \lambda \in \Lambda\}.$$

It follows that, for this x, $\psi(x) = \{\lambda(x): \lambda \in \Lambda\}$. Hence, for all $x \in X$, $\psi(x) \leq \sup \{\lambda(x): \lambda \in \Lambda\}$. Since $\psi \geq \sup \{\lambda: \lambda \in \Lambda\}$, as noted earlier, it follows that $\psi = \sup \{\lambda: \lambda \in \Lambda\}$. Now let λ_1 and λ_2 be in Λ. There exist functions ϕ_1 and $\phi_2 \in \Phi$ such that $\lambda_1 = \min \{\psi, \phi_1\}$, $\lambda_2 = \min \{\psi, \phi_2\}$. Furthermore, there exists functions g_1 and $g_2 \in G$ such that $\phi_1 \leq g_1$ and $\phi_2 \leq g_2$. Since G is directed upward, there exists a function $g \in G$ such that $g_1 \leq g$ and $g_2 \leq g$. Hence, since $g = \sup \{\phi: \phi \in \Phi_g\}$, there is a $\phi \in \Phi_g$ such that $\phi_1 \leq g_1 \leq \phi$ and $\phi_2 \leq g_2 \leq \phi$. Then the element $\lambda = \min \{\psi, \phi\}$ is such that $\lambda \geq \lambda$, and $\lambda \geq \lambda_2$. Thus Λ is an upward directed subset of $C_r^{o+}(X)$. Since $\psi = \sup \{\lambda: \lambda \in \Lambda\}$, $\mu(\psi) = \sup \{\mu(\lambda): \lambda \in \Lambda\}$ by the first part of the argument. But

$$\sup \{\mu(\lambda): \lambda \in \Lambda\} \leq \sup \{\mu(\phi): \phi \in \Phi\} \leq$$

$$\sup \{\mu^*(\phi): \phi \in \Phi\},$$

since $\mu(\phi) = \mu^*(\phi)$. Furthermore,

$$\sup \{\mu^*(\phi): \phi \in \Phi\} \leq \sup \{\mu^*(g): g \in G\}.$$

Hence $\mu(\psi) \leq \sup \{\mu^*(g): g \in G\}$ for all $\psi \in C_r^{o+}(X)$ such that $\psi \leq f$. Hence, by definition of $\mu^*(f)$, $\mu^*(f) \leq \sup \{\mu^*(g): g \in G\}$. Combining this with (1), we have $\mu^*(f) = \sup\{\mu^*(g): g \in G\}$. $\square$

Theorem 3.14. Let f_1 and f_2 be in $LS^+_{r*}(X)$. Then

$$\mu^*(f_1 + f_2) = \mu^*(f_1) + \mu^*(f_2).$$

Proof. Let

$$\Phi_i = \{\phi \in C_r^{o+}(X) : \phi \le f_i\}, \quad i = 1, 2.$$

Then $\sup \Phi_1 + \sup \Phi_2 = f_1 + f_2$ by Theorem 2.11. Hence,

$$f_1 + f_2 = \sup \Phi_1 + \sup \Phi_2 = \sup \{\Phi_1 + \Phi_2 : \phi_1 \in \Phi_1 \text{ and } \phi_2 \in \Phi_2\}.$$

It may happen that $f_1 + f_2 \in \{\phi_1 + \phi_2 : \phi_1 \in \Phi_1 \text{ and } \phi_2 \in \Phi_2\}$ in which case the conclusion is obvious. If $f_1 + f_2 \notin \{\phi_1 + \phi_2 : \phi_1 \in \Phi_1$ and $\phi_2 \in \Phi_2\}$, then this set is directed upward. Hence, by Theorem 3.13,

$$\begin{aligned}\mu^*(f_1 + f_2) &= \sup \{\mu(\phi_1 + \phi_2) : \phi_1 \in \Phi_1 \text{ and } \phi_2 \in \Phi_2\} = \\ &\sup \{\mu(\phi_1) + \mu(\phi_2) : \phi_1 \in \Phi_1 \text{ and } \phi_2 \in \Phi_2\} = \\ &\sup \{\mu(\phi_1) : \phi_1 \in \Phi_1\} + \sup\{\mu(\phi_2) : \phi_2 \in \Phi_2\} \\ &= \mu^*(f_1) + \mu^*(f_2).\end{aligned}$$

□

Theorem 3.15. For every sequence $(f_i)_{i\ge 1}$ in $LS^+_{r*}(X)$,

$$\mu^*\left(\sum_{i=1}^{\infty} f_i\right) = \sum_{i=1}^{\infty} \mu^*(f_i).$$

Proof. By Theorem 2.7, $\sum_{i=1}^{\infty} f_i$ is in $LS^+_{r^*}(X)$. Let J be any finite subset of the positive integers. Define $F_J = \sum_{j\varepsilon J} f_j$. By Theorem 3.14, we have $\mu^*(F_J) = \sum_{i\varepsilon J} \mu^*(f_i)$. Given F_i and F_j, we have that $F_{1+\max\{i,j\}} \geq F_i$ and $F_{1+\max\{i,j\}} \geq F_j$ where $F_1 = f_1$. Hence the family $\{F_j\}$ is directed upward. Since $\sup_{j\geq 1} \{F_j\} = \sum_{i=1}^{\infty} f_i$, the conclusion follows by Theorem 3.13. ∎

We shall now extend μ^* so that it acts on all positive extended-real-valued functions.

Definition 3.16. Let f be a positive extended-real-valued function on X. We define

$$\mu^*(f) = \inf \{\mu^*(g)\colon g \in LS^+_{r^*}(X),\ \ g \geq f\}.$$

Note that the set in Definition 3.16 is never empty since the constant function $g = \infty$ is in $LS^+_{r^*}(X)$, and $g \geq f$. It is obvious that μ^* agrees with the original μ^* on $LS^+_{r^*}(X)$. It is also obvious that if $f \leq g$, then $\mu^*(f) \leq \mu^*(g)$, and, if $\lambda \geq 0$, then $\mu^*(\lambda f) = \lambda\mu^*(f)$.

Theorem 3.17. If f_1 and f_2 are two positive extended-real-valued functions defined on X, then

$$\mu^*(f_1 + f_2) \leq \mu^*(f_1) + \mu^*(f_2).$$

Proof. If g_1 and g_2 are in $LS^+_{r*}(X)$ and such that $f_1 \le g_1$ and $f_2 \le g_2$, then

$$\mu^*(f_1 + f_2) \le \mu^*(g_1 + g_2) = \mu^*(g_1) + \mu^*(g_2)$$

by Theorem 3.14. Now

$$\begin{aligned}\mu^*(f_1) + \mu^*(f_2) = {} & \inf \{\mu^*(g_1)\colon g_1 \in LS^+_{r*}(X) \text{ and } g_1 \ge f_1\} + \\ & \inf \{\mu^*(g_2)\colon g_2 \in LS^+_{r*}(X) \text{ and } g_2 \ge f_2\} = \\ & \inf \{\mu^*(g_1) + \mu^*(g_2)\colon g_1 \text{ and } g_2 \in \\ & LS^+_{r*}(X),\ g_1 \ge f_1 \text{ and } g_2 \ge f_2\}.\end{aligned}$$

Hence $\mu^*(f_1 + f_2) \le \mu^*(f_1) + \mu^*(f_2)$. ▯

Definition 3.18. If f is an extended-real-valued function defined on X, we set

$$N(f) = \mu^*(|f|).$$

Definition 3.19. We denote by $F(X, \mu)$ the set of all real-valued functions on X such that $N(f) < \infty$ for all $f \in F(X, \mu)$.

Theorem 3.20. The set $F(X, \mu)$ is a vector space over R and the mapping $f \to N(f)$ is a pseudo-norm on $F(X), \mu)$.

Proof. If f and g are in $F(X, \mu)$, then

$$N(f + g) = \mu^*(|f + g|) \leq \mu^*(|f| + |g|) \leq \mu^*(|f|) + \mu^*(|g|)$$

$$= N(f) + N(g) < \infty$$

which shows that $f + g \in F(X, \mu)$ and that N satisfies the triangle inequality. Now

$$N(\lambda f) = \mu^*(|\lambda f|) = |\lambda|\cdot\mu^*(|f|) = |\lambda|\cdot N(f) < \infty.$$

It has now been established that $F(X, \mu)$ is a vector space over R. If $f = 0$, then $N(f) = \mu(0) = 0$. Hence we have that N is a pseudo-norm on $F(X, \mu)$.

Definition 3.20. We denote by $L(X, \mu)$ the closure of $C_r^O(X)$ in $F(X, \mu)$ with respect to the topology induced by N.

Theorem 3.21. The set $L(X, \mu)$ is a vector space over R and the mapping $f \to N(f)$ is a pseudo-norm on $L(X, \mu)$.

Proof. Let f_1 and f_2 be in $L(X, \mu)$. Let $\varepsilon > 0$ be given. If f_1 and $f_2 \in C_r^O(X)$, then obviously $f_1 + f_2 \in L(X, \mu)$. Hence, suppose neither f_1, f_2, nor $f_1 + f_2$ are in $C_r^O(X)$. Then there exist functions g_1 and g_2

in $C_r^o(X)$ such that $N(f_1 - g_1) < \varepsilon/2$ and $N(f_2 - g_2) < \varepsilon/2$. Then $N(f_1 + f_2 - g_1 - g_2) \leq N(f_1 - g_1) + N(\varepsilon_2 - g_2) < \varepsilon$. It follows that $f_1 + f_2 \in L(X, \mu)$. If at least one of the functions f_1, f_2 is in $C_r^o(X)$, a similar argument applies. Thus $L(X, \mu)$ is algebraically closed under addition.

If $f \in L(X, \mu)$ and if λ is a real number, we shall show that $\lambda \cdot f \in L(X, \mu)$. If either $f \in C_r^o(X)$ or $\lambda \cdot f \in C_r^o(X)$, then the conclusion is obvious. Hence assume that $f \notin C_r^o(X)$ and that $\lambda f \notin C_r^o(X)$. Then $\lambda \neq 0$. Furthermore, there exists a function $g \in C_r^o(X)$ such that $N(f - g) < \varepsilon/|\lambda|$. Then $N(\lambda f - \lambda g) < \varepsilon$. Since $\lambda g \in C_r^o(X)$, we have that λf is in the closure of $C_r^o(X)$; that is, $\lambda g \in L(X, \mu)$. Hence $L(X, \mu)$ is a vector space. The mapping N is a pseudo-norm on $L(X, \mu)$ since it is a pseudo-norm on $F(X, \mu)$. $\square$

<u>Theorem 3.22</u>. If the functions f and g are in $L(X, \mu)$, then the functions $|f|$, $\sup\{f, g\}$, and $\inf\{f, g\}$ are in $L(X, \mu)$.

<u>Proof</u>. Let (ϕ_n) be a sequence of functions in $C_r^o(X)$ such that $N(f - \phi_n) \to 0$. By standard properties of a pseudo-norm, we have that $N(|f| - |\phi_n|) \leq N(f - \phi_n)$. Hence $N(|f| - |\phi_n|) \to 0$. It follows that $|f|$ is in $L(X, \mu)$, since $(|\phi_n|)$ is a sequence in $C_r^o(X)$, noting Theorem 3.2. It is easily verified that $\sup\{f, g\} = 1/2(f + g + |f - g|)$, and $\inf\{f, g\} = 1/2(f + g - |f - g|)$. It follows that, since $L(X, \mu)$ is a vector space, that $\sup\{f, g\}$ and $\inf\{f, g\}$ are in $L(X, \mu)$. $\square$

Theorem 3.23. There exists a unique positive linear functional on $L(X, \mu)$ which is an extension of μ and is continuous on $L(X, \mu)$ with respect to N.

Proof. Let f be a function in $L(X, \mu)$. Let f^+ be the function $\sup \{f, 0\}$ and let f^- be the function $\sup \{-f, 0\}$. Then f^+ and f^- are both in $L(X, \mu)$ and are non-negative. We define

$$\mu(f) = \mu^*(f^+) - \mu^*(f^-).$$

Since $f = f^+ - f^-$, it is obvious in view of Theorem 3.11 that this definition of μ agrees with the original definition on $C_r^0(X)$. We note that μ may be thought of as an extension of μ^* to $L(X, \mu)$ in the sense of Theorem 3.5. We note also that the conclusion of Theorem 3.5 remains valid if $C_r^0(X)$ is replaced by any vector lattice. We have shown that $L(X, \mu)$ is a vector lattice in Theorem 3.22; hence μ is a positive linear functional on $L(X, \mu)$. We note also that μ is order preserving on $L(X, \mu)$; that is, given $f \le g$ we have $\mu(f) \le \mu(g)$. Hence, if $f \in L(X, \mu)$,

$$-N(f) = \mu^*(|f|) = -\mu(|f|) \le \mu(f) \le \mu(|f|) = N(f).$$

Thus $|\mu(f)| \le N(f)$ for all $f \in L(X, \mu)$. It follows that if (f_n) is a sequence of functions in $L(X, \mu)$ such that $N(f_n) \to 0$, we have $|\mu(f_n)| \to 0$. Thus μ is continuous at the zero function. Hence μ is continuous on the entire space $L(X, \mu)$. Suppose that μ' is a continuous positive linear functional which agrees with μ on $C_r^o(X)$. Let f be in $L(X, \mu)$, and suppose $f \notin C_r^o(X)$. There exists a sequence of functions (ϕ_n) in $C_r^o(X)$ such that $N(f - \phi_n) \to 0$. Hence, given $\varepsilon > 0$, there exists an integer P_ε such that for all $n \geq P_\varepsilon$, $|\mu(f) - \mu(\phi_n)| < \varepsilon/2$, and $|\mu'(f) - \mu'(\phi_n)| < \varepsilon/2$. Then

$$|\mu(f) - \mu'(f)| \leq |\mu(f) - \mu(\phi_n)| + |\mu'(\phi_n) - \mu'(f)| < \varepsilon.$$

Hence $|\mu(f) - \mu'(f)| = 0$. Thus μ is a unique extension. ▯

The number $\mu(f)$ will be called *the integral of f with respect to* μ or, if no confusion can arise, the *integral of f*. Other notations that will be used for $\mu(f)$ are $\int f$ and $\int f\, d\mu$.

Theorem 3.24. Let $f \in LS_r^+(X)$. Then $f \in L(X, \mu)$ if and only if $\mu^*(f) < \infty$.

Proof. Suppose that $f \in L(X, \mu)$. Then $\infty > N(f) = \mu^*(f)$. Now assume that $\mu^*(f) < \infty$. Then $f \in F(X, \mu)$ since f is real-valued. Let $\varepsilon > 0$ be given.

There exists a function $\phi \in C_r^{o+}(X)$ such that $\phi \leq f$ and $\mu(\phi) > \mu^*(f) - \varepsilon$. Since $f = \phi + (f - \phi)$, we have, by Theorem 3.14, $\mu^*(f) = \mu(\phi) + \mu^*(f - \phi)$. Hence $\mu^*(f - \phi) = \mu^*(f) - \mu(\phi) < \varepsilon$. Thus $f \in L(X, \mu)$, since $N(f - \phi) = \mu^*(f - \phi)$. ▯

Theorem 3.25. If f is a non-negative function in $L(X, \mu)$, and if $c > 0$, then the function g defined by $g = \min\{f, c\}$ is integrable.

Proof. Let $\varepsilon > 0$ be given. From the definition of $\mu^*(f)$ and the fact that f is integrable, there exists a function ϕ in $LS^+_{r*}(X)$ such that $\phi \geq f$ and $\mu^*(\phi) < \mu^*(f) + \varepsilon < \infty$. Hence $\mu^*(\phi) < \infty$ and it follows that ϕ is integrable. Since f and ϕ are non-negative, we have that $N(\phi) < N(f) + \varepsilon$. Hence

$$N(\phi - f) = \mu^*(\phi - f) = \mu(\phi - f) = \mu(\phi) - \mu(f) < \varepsilon.$$

Now we set $\psi(x) = \min \{\phi(x), c\}$ for all $x \in X$. The constant function c is lower semicontinuous by definition, and hence ψ is lower semicontinuous by Theorem 2.4. Furthermore, since $\psi \leq \phi$, we have $N(\psi) \leq N(\phi) < \infty$. Thus ψ is integrable. Note also that $\psi \geq g$. We shall show that $\psi - g \leq \phi - f$. Let $x \in X$. If $f(x) \geq c$, then

$$\psi(x) - g(x) = c - c = 0 \leq \phi(x) - f(x).$$

If $f(x) < c$, then

$$\psi(x) - g(x) = \min \{\phi(x), c\} - f(x) \leq \phi(x) - f(x).$$

Hence in either case, $\psi(x) - g(x) \leq \phi(x) - f(x)$ for all x in X. It follows that $N(\psi - g) \leq N(\phi - f) < \varepsilon$. Hence g is integrable by definition of $L(X, \mu)$. $\square$

Theorem 3.26. Let G be an upward directed denumerable family of positive extended-real-valued functions defined on X. If $f = \sup\{g: g \in G\}$ then

$$\mu^*(f) = \sup\{\mu^*(g): g \in G\}.$$

Proof. We shall first suppose that G is an increasing sequence. Since $g_n \le f$ for all $n \ge 1$, we have $\mu^*(g_n) \le \mu^*(f)$, for all $n \ge 1$. Hence

$$\text{(1)} \qquad \sup\{\mu^*(g_n): n \ge 1\} \le \mu^*(f).$$

If, for some n, $\mu^*(g_n) = \infty$, the conclusion is obvious. Hence, we may assume that $\mu^*(g_n) < \infty$ for all n. It follows that there exists a function $G_n \in LS^+_{r*}(X)$ such that $\mu^*(G_n) \le \mu^*(g_n) + \varepsilon/2^n$ for each $n \ge 1$. Define $H_n = \sup\{G_1, \ldots, G_n\}$. Then $H_n \in LS^+_{r*}(X)$. Moreover, $H_n \le H_{n+1}$ and $H_n \ge g_n$ for all $n \ge 1$. Note that

$$\text{(2)} \qquad H_{n+1} = \sup\{G_1, \ldots, G_{n+1}\} = \sup\{H_n, G_{n+1}\}.$$

Since $g_n \le H_n$ and $g_n \le g_{n+1} \le G_{n+1}$, then $g_n \le \inf\{H_n, G_{n+1}\}$ for all $n \ge 1$. Now

$$\sup\{H_n, G_{n+1}\} + \inf\{H_n, G_{n+1}\} = H_n + G_{n+1}.$$

Since μ^* is additive on $LS^+_{r*}(X)$, then

$$\mu^*(\sup\{H_n, G_{n+1}\}) + \mu^*(\inf\{H_n, G_{n+1}\}) = \mu^*(H_n) + \mu^*(G_{n+1}).$$

Thus, by (2),

$$\mu^*(H_{n+1}) + \mu^*(\inf \{H_n, G_{n+1}\}) = \mu^*(H_n) + \mu^*(G_{n+1}).$$

Hence, since $\inf \{H_n, G_{n+1}\} \geq g_n$ and $\inf \{H_n, G_{n+1}\} \in LS^+_{r*}(X)$,

$$\mu^*(H_{n+1}) + \mu^*(g_n) \leq \mu^*(H_n) + \mu^*(G_{n+1}).$$

Hence

$$\mu^*(H_{n+1}) \leq \mu^*(H_n) + \mu^*(G_{n+1}) - \mu^*(g_n) \leq$$

$$\mu^*(H_n) + \mu^*(g_{n+1}) + \varepsilon/2^{n+1} - \mu^*(g_n).$$

Thus

$$\mu^*(H_k) - \mu^*(H_{k-1}) \leq \mu^*(g_k) - \mu^*(g_{k-1}) + \varepsilon/2^k,$$

for $k \geq 2$. Hence

$$\mu^*(H_n) - \mu^*(H_1) \leq \mu^*(g_n) - \mu^*(g_1) + \sum_{k=2}^{n} \frac{\varepsilon}{2^k} .$$

Hence, noting that $H_1 = G_1$,

$$\mu^*(H_n) \le \mu^*(g_n) + \mu^*(H_1) - \mu^*(g_1) + \sum_{k=2}^{n} \frac{\varepsilon}{2^k} \le$$

$$\mu^*(g_n) + \sum_{k=1}^{n} \frac{\varepsilon}{2^k} < \mu^*(g_n) + \varepsilon.$$

Let $H = \sup \{H_n: n \ge 1\} = \sup \{G_n: n \ge 1\}$. Then $f \le H$, since $g_n \le G_n$ for all $n \ge 1$ and $f = \sup \{g_n: n \ge 1\}$. Consequently, by the monotone property of μ^* and by Theorem 3.13,

$$\mu^*(f) \le \mu^*(H) = \sup \{\mu^*(H_n): n \ge 1\} < \sup \{\mu^*(g_n): n \ge 1\} + \varepsilon.$$

Thus $\mu^*(f) \le \sup \{\mu^*(g_n): n \ge 1\}$. Noting (1), we have

$$\mu^*(f) = \sup \{\mu^*(g_n): n \ge 1\}.$$

If G is a denumerable directed set (g_i), we define a sequence of integers and functions as follows. Let $n_1 = 1$ and let $g_{n_1} = g_1$. Assume now that k such integers and functions have been defined. Then there exists a function $g_i \in G$ such that $g_i \ge g_{k+1}$ and $g_i \ge g_{n_k}$. Define $k + 1 = i$ and $g_{n_{k+1}} = g_i$. Then (g_{n_k}) is an increasing subsequence of the original sequence (g_i). Now

$$\{g_{n_k}: k \ge 1\} \subseteq \{g_i: i \ge 1\}$$

and hence

$$\sup \{g_{n_k} : k \geq 1\} \leq \sup \{g_i : i \geq 1\}.$$

But, for any function $g_i \in G$, there exists an index k such that $g_{n_k} \geq g_i$. Thus $\sup \{g_{n_k} : k \geq 1\} \geq \sup \{g_i : i \geq 1\}$, and hence

$$\sup \{g_{n_k} : k \geq 1\} = \sup \{g_i : i \geq 1\} = f.$$

A similar argument shows that

$$\sup \{\mu^*(g_{n_k}) : k \geq 1\} = \sup \{\mu^*(g_i) : i \geq 1\}.$$

Hence, by the first part of the proof,

$$\mu^*(f) = \sup \{\mu^*(g_i) : i \geq 1\}. \qquad \square$$

The above result is usually known as Fatou's theorem, and the following two results are consequences of it.

Theorem 3.27. Let $\{f_n\}$ be a sequence of positive extended-real-valued functions on X. Then

$$\mu^*\left(\sum_{n=1}^{\infty} f_n\right) \leq \sum_{n=1}^{\infty} \mu^*(f_n).$$

Proof. Define $g_n = \sum_{i=1}^{n} f_i$. Then by the corresponding conclusion for the finite case and Theorem 3.26,

$$\mu^*\left(\sum_{n=1}^{\infty} f_n\right) = \mu^*(\sup\{g_n : n \geq 1\}) = \sup\{\mu^*(g_n) : n \geq 1\}$$

$$\leq \sup\left\{\sum_{i=1}^{n} \mu^*(f_i) : n \geq 1\right\} = \sum_{n=1}^{\infty} \mu^*(f_n). \qquad \square$$

The following consequence of Theorem 3.26 is known as Fatou's Lemma.

<u>Theorem 3.28</u>. Let (f_n) be a sequence of positive extended-real-valued functions defined on X. Then $\mu^*(\underline{\lim}\, f_n) \leq \underline{\lim}\, \mu^*(f_n)$.

<u>Proof</u>. Define $g_n = \inf\{f_n, f_{n+1}, \ldots\}$, $n \geq 1$. Then the sequence (g_n) is increasing, and $\sup(g_n) = \underline{\lim}\, f_n$ by definition of $\underline{\lim}\, f_n$. Thus

(1) $$\mu^*(g_n) \leq \inf\{\mu^*(f_n), \mu^*(f_{n+1}), \ldots\}$$

since $\mu^*(g_n) \leq \mu^*(f_i)$ for all $i \geq n$.
By Theorem 3.26,

$$\mu^*(\underline{\lim}\, f_n) = \mu^*(\sup_{n\geq 1} g_n) = \sup_{n\geq 1} \mu^*(g_n) \leq \underline{\lim}\, \mu^*(f_n)$$

since, from (1), $\sup_{n\geq 1} \mu^*(g_n) \leq \underline{\lim}\, \mu^*(f_n)$. $\square$

<u>Definition 3.29</u>. Let f be an extended-real-valued function defined on X. Then f is called a *null function* if and only if $\mu^*(|f|) = 0$.

If f is a real-valued null function, then f is in $L(X, \mu)$ since the sequence $\{0, 0, \ldots\}$ converges to f, and each member of the sequence is in $C_r^o(X)$.

Theorem 3.30. Let f and g be extended-real-valued functions defined on X and let (f_n) be a sequence of functions defined on X. Then:

a. If $0 \leq g \leq f$ and f is a null function, then g is a null function.

b. If λ is a real number and if f is a null function, then λf is a null function.

c. If $f_n \geq 0$ for each n, and if $f = \sum_{n=1}^{\infty} f_n$, then f is a null function if and only if each f_n is a null function.

d. If (f_n) is a sequence of non-negative functions, then the function $f = \sup_{n\geq 1} f_n$ is a null function if and only if each f_n is a null function.

Proof. Parts (a) and (b) are obvious from the definition of null functions and basic properties of μ^*. If $f = \sum_{n=1}^{\infty} f_n$, and each f_n is a null function, then

$$0 \leq \mu^*(|f|) \leq \sum_{n=1}^{\infty} \mu^*(|f_n|) = 0$$

by Theorem 3.27. Hence f is a null function. Clearly each f_n is a null function if f is a null function. Now $\sup_{n\geq 1} f_n \leq \sum_{n=1}^{\infty} f_n$. Hence $\sup_{n\geq 1} f_n$ is a null function if each f_n is null, by parts (a) and (c). Clearly the converse of this assertion is also true. ▯

Definition 3.31. Let A be a subset of X. If the characteristic function χ_A is a null function, then A is said to be a *null set*.

Theorem 3.32. Suppose A and B are subsets of X. If $A \subsetneq B$ and B is a null set, then A is a null set. If $A = \bigcup_{n\geq 1} A_n$, then A is a null set if and only if each A_n is a null set.

The assertions are immediate consequences of Theorem 3.31.

Definition 3.33. A property of points of X is said to hold *almost everywhere on X* if the set of points for which the property does not hold is a null set.

If P is a property of points of X, then we shall use the abbreviation P a.e. to stand for the sentence, "P holds almost everywhere on X."

Theorem 3.34. Let f be an extended-real-valued function defined on X. Then f is a null function if and only if $f(x) = 0$ a.e. on X.

Proof. Suppose $f(x) = 0$ a.e. on X. Let $D = \{x\colon f(x) \neq 0\}$. Then $0 \leq |f| \leq \sup \{n\, \chi_D\colon n \geq 0\}$. Hence by Theorem 3.30, $|f|$ is a null function since $n\chi_D$ is a null function for each n. Conversely, suppose that f is a null function. Then

$$0 \leq \chi_D \leq \sup \{n|f|\colon n = 0, 1, \ldots\}.$$

Hence, by Theorem 3.30, χ_D is a null function since $n|f|$ is a null function. $\square$

Theorem 3.35. If f and g are non-negative extended-real-valued functions on X and if $f(x) \leq g(x)$ a.e. on X, then $\mu^*(f) \leq \mu^*(g)$.

Proof. Let $D = \{x: f(x) > g(x)\}$. Then D is a null set. Define

$$h(x) = \begin{cases} \infty, & x \in D \\ 0, & x \notin D \end{cases}$$

Then $h = \sup_{n\geq 1} \{n\chi_D: n = 1, 2, \ldots\}$. Thus h is a null function. Now $f \leq g + h$. Hence

$$\mu^*(f) \leq \mu^*(g) + \mu^*(h) = \mu^*(g). \qquad \square$$

Theorem 3.36. If f and g are positive extended-real-valued functions on X and $f(x) = g(x)$ a.e. on X, then $\mu^*(f) = \mu^*(g)$.

Proof. Note that $f(x) \leq g(x)$ a.e. and $g(x) \leq f(x)$ a.e. Hence by Theorem 3.35, $\mu^*(f) = \mu^*(g)$. $\square$

Theorem 3.37. Let f be a positive extended-real-valued function on X such that $\mu^*(f) < \infty$. Then $f(x) < \infty$ a.e. on X.

Proof. Let $D = \{x: f(x) = \infty\}$. Then for every positive integer $n \geq 0$, $n\chi_D \leq f$. Hence $n\mu^*(\chi_D) \leq \mu^*(f) < \infty$. Thus $\mu^*(\chi_D) = 0$. Hence D is a null set. $\square$

Theorem 3.38. Let (f_n) be a monotone increasing sequence of functions in $L(X, \mu)$. Then the limit function f is in $L(X, \mu)$ if f is real valued and if $\lim_{n\to\infty} \mu(f_n) < \infty$, in which case $\mu(f) = \lim_{n\to\infty} \mu(f_n)$.

Proof. We first consider the case in which each f_n is non-negative. Then $\mu^*(f_n) = \mu(f_n)$ for all n. Theorem 3.26 shows that $\mu^*(f) = \lim_{n\to\infty} \mu(f_n)$. It remains to show that if $\mu^*(f) < \infty$, then f is in $L(X, \mu)$. Let $\varepsilon > 0$ be given. Using the definition of $\mu^*(f)$, there exists a function $\bar{\phi} \in LS^+_{r^*}(X)$ such that $\bar{\phi} \geq f$ and $\mu^*(\bar{\phi}) < \mu^*(f) + \varepsilon < \infty$. We now consider the function ϕ defined by

$$\phi(x) = \begin{cases} \bar{\phi}(x), & \bar{\phi}(x) < \infty \\ \\ 0, & \bar{\phi}(x) = \infty . \end{cases}$$

Then $\phi(x) = \bar{\phi}(x)$ a.e. on X and hence $\mu^*(\phi) = \mu^*(\bar{\phi})$. It is shown in the proof of Theorem 3.24 that there exist a sequence (ϕ_n) in $C_r^{o+}(X)$ such that $\lim_{n\to\infty} N(\bar{\phi} - \phi_n) \to 0$. Noting that $N(\phi - \phi_n) \leq N(\bar{\phi} - \phi_n)$ since $|\phi(x) - \phi_n(x)| \leq |\bar{\phi}(x) - \phi_n(x)|$ a.e., we have that $N(\phi - \phi_n) \to 0$. Hence ϕ is in $L(X, \mu)$, and $\phi - f_n$ is integrable for each $n \geq 1$. Since $\phi \geq f \geq f_n$, $\phi - f_n$ is non-negative. Thus

$$\mu^*(\phi - f_n) = \mu(\phi - f_n) = \mu(\phi) - \mu(f_n).$$

Since $f - f_n \geq 0$,

$$N(f - f_n) = \mu^*(f - f_n) \leq \mu^*(\phi - f_n) = \mu(\phi) - \mu(f_n).$$

Hence $\lim_{n\to\infty} N(f - f_n) \leq \mu(\phi) - \mu^*(f) < \varepsilon$. Let $k = \lim_{n\to\infty} N(f - f_n)$. There exists an integer n such that $|N(f - f_n) - k| = N(f - f_n) - k < \varepsilon$. Thus $N(f - f_n) < k + \varepsilon < 2\varepsilon$. It follows that f in integrable.

If (f_n) is an arbitrary sequence satisfying the hypotheses, then consider the sequence defined by $g_n = f_n - f_1$. Then (g_n) is an increasing sequence of non-negative functions, and $\lim_{n\to\infty} g_n = \lim_{n\to\infty} f_n - f_1$. Since $\lim_{n\to\infty} \mu(g_n) = \lim_{n\to\infty} \mu(f_n) - \mu(f_1) < \infty$, we have that the limit function g is integrable and $g + f_1 = f$. Hence f is integrable. Furthermore,

$$\mu(f) = \mu(g) + \mu(f_1) = \lim_{n\to\infty} \mu(g_n) + \mu(f_1) = \lim_{n\to\infty} \mu(g_n + f_1) = \lim_{n\to\infty} \mu(f_n). \square$$

The above result is usually known as the *monotone convergence theorem*.

Definition 3.39. Let A be a subset of X. The number $\mu^*(\chi_A)$ will be denoted by $\mu^*(A)$ and called the *outer measure* of A.

Theorem 3.40. The set function defined by $\mu^*(A) = \mu^*(\chi_A)$ for all $A \subseteq X$ is an outer measure on the subsets of X.

Proof: Clearly $0 \leq \mu^*(A) \leq \infty$ for all $A \subseteq X$. If $A \subseteq B$, then $\chi_A \leq \chi_B$ and hence $\mu^*(A) \leq \mu^*(B)$. If $A = \bigcup_{n\geq 1} A_n$, then

$$\chi_A = \sup_{n\geq 1}\{\chi_{A_n}\} \leq \sum_{n=1}^{\infty} \chi_{A_n},$$

and hence

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

It follows that μ^* is an outer measure on the subsets of X. ▯

We postpone the development of a *measure* for the subsets of X until Chapter IV.

We now introduce the concept of $L_p(X, \mu)$ for $1 \leq p < \infty$.

Definition 3.41. Let p be a positive number, $p > 1$. For any extended-real-valued function f we define $N_p(f) = \mu^*(|f|^p)^{1/p}$.

Definition 3.42. We denote by $F_p(X, \mu)$ the set of real-valued functions on which N_p is real-valued.

When $p = 1$, the above definitions yield N and $F(X, \mu)$ as used before. The classical Hölder and Minkowski inequalities hold for N_p, and the proofs are in no way different from the classical ones [6].

It then follows that $F_p(X, \mu)$ is a vector space and N_p is a pseudo-norm on $F_p(X, \mu)$.

Definition 3.43. The closure of the set $C_r^o(X)$ in $F_p(X, \mu)$ with respect to the topology induced by N_p will be denoted by $L_p(X, \mu)$.

As before, one can show that $L_p(X, \mu)$ is a vector space and that N_p is a pseudo-norm on $L_p(X, \mu)$. It is also the case that $L_p(X, \mu)$ is a *complete* space.

Theorem 3.44. Let (f_n) be a Cauchy sequence of functions in $\mathcal{F}_p(X, \mu)$. Then there exist a subsequence (f_{n_k}) such that $\lim_{k\to\infty} f_{n_k}(x)$ exist a.e. and is real-valued. Furthermore, if $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ and f is real-valued, then $N_p(f_{n_k} - f) \to 0$. In particular, $\mathcal{F}_p(X, \mu)$ is complete.

The proof is omitted [6].

It is easily shown that a closed subspace of a complete pseudo-normed vector space is complete. Hence $\mathcal{L}_p(X, \mu)$ is complete. Note that $\mathcal{L}_p(X, \mu)$ is not a normed vector space, but the set $L_p(X, \mu) = \mathcal{L}_p(X, \mu)/R$ where $R = \{(f, g): N_p(f - g) = 0\}$ is a normed vector space; it is, in fact, a *Banach space*. The following results are consequences of Theorem 3.44.

Theorem 3.45. If f and f^* are real-valued functions and if (f_n) is a sequence of real-valued functions such that $f_n(x) \to f^*(x)$ a.e. on X and $N_1(f - f_n) \to 0$, then $f(x) = f^*(x)$ a.e. on X.

Proof. Let $A = \{x: f(x) \neq f^*(x)\}$. By Theorem 3.44 there exists a subsequence $\{f_{n_j}\}$ such that $\lim_{j\to\infty} f_{n_j}(x) = f(x)$ a.e. on X. Let $B = \{x: \lim_{j\to\infty} f_{n_j}(x) \neq f^*(x)\}$, and let $C = \{x: \lim_{j\to\infty} f_{n_j}(x) \neq f(x)\}$. If $x \notin B \cup C$, then clearly $x \notin A$. Hence $A \subseteq B \cup C$. But B and C are null sets. Hence A is a null set. $\square$

Theorem 3.46. Let f be a real-valued function on X. If f in $\mathcal{L}(X, \mu)$, there exists an N-Cauchy sequence (f_n) with each $f_n \in C_r^o(X)$ such that $f_n(x) \to f(x)$ a.e.

Proof. Since $L(X, \mu)$ is the closure of $C_r^o(X)$ in $F(X, \mu)$, there exists a sequence of functions (f_n^*) in $C_r^o(X)$ such that $N(f - f_n^*) \to 0$. This implies that the sequence (f_n^*) is N-Cauchy. Hence, by Theorem 3.44 there exists a subsequence (f_n) of (f_n^*) such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. ▯

Theorem 3.47. Let f be a real-valued function on X. Then if there exists an N-Cauchy sequence of functions (f_n) in $L(X, \mu)$ such that $f_n \to f$ a.e., then f is in $L(X, \mu)$.

Proof. Since $L(X, \mu)$ is complete, there exists a function f^* such that $N(f_n - f^*) \to 0$. By Theorem 3.44 there exists a subsequence (f_n^*) of (f_n) such that $f_n^*(x) \to f(x)$ a.e. on X. By Theorem 3.45, $f(x) = f^*(x)$a.e. Hence, by Theorem 3.35, f is in $L(X, \mu)$ since f is real-valued. ▯

Theorem 3.48. If f and g are two bounded functions in $L(X, \mu)$, then the function $h = fg$ is in $L(X, \mu)$.

Proof. By Theorem 3.46, there exist N-Cauchy sequences (f_n^*) and (g_n^*) in $C_r^o(X)$ such that $\lim f_n^*(x) = f(x)$a.e. and $\lim g_n^*(x) = g(x)$a.e. Let $\varepsilon > 0$ be given, and let M be a bound on both f and g. Choose

$$f_n(x) = \begin{cases} f_n^*(x), & |f_n^*(x)| \leq M \\ M, & |f_n^*(x)| > M \end{cases}$$

Then each f_n is in $C_r^o(X)$ and $f_n(x) \to f(x)$a.e. on X. We obtain in a similar way a bounded sequence (g_n) in $C_r^o(X)$ with $g_n(x) \to g(x)$a.e. on X.

Note that $\{f_n\}$ and $\{g_n\}$ are N-Cauchy sequences. Choose P_1 and P_2 to be positive integers such that for all n, $m \geq P_1$, $N(f_n - f_m) < \varepsilon/2M$, and for all n, $m \geq P_2$, $N(g_n - g_m) < \frac{\varepsilon}{2M}$. Choose $P = \max\{P_1, P_2\}$. Then, for all n, $m \geq P$, $N(f_m g_m - f_m g_m) = N(f_n g_n - f_n g_m + f_n g_m - f_m g_m) \leq MN(g_n - g_m) + MN(f_n - f_m) < \varepsilon$. Thus $(f_n g_n)$ is an N-Cauchy sequence in $L(X, \mu)$. Now choose P so that for all $n \geq P$, $N(f_n - f) < \frac{\varepsilon}{2M}$, and $N(g_n - g) < \varepsilon/2M$. Then

$$N(f_n g_n - fg) = N(f_n g_n - fg_n + fg_n - fg) \leq MN(f_n - f) + MN(g_n - g) < \varepsilon.$$

Hence, by Theorem 3.47, fg is in $L(X, \mu)$. ∎

CHAPTER IV

MEASURE THEORY

In this chapter we show how the integral developed in the preceding chapter yields a measure on a family of subsets of X. We also investigate under certain conditions some of the relationships between two distinct measures.

<u>Definition 4.1</u>. Let A be a subset of X. Then A is said to be *integrable* if and only if χ_A, the characteristic function of A, is in $L(X, \mu)$.

<u>Theorem 4.2</u>. Let (A_n) be a sequence of integrable sets.

1. If $A_{n+1} \supsetneq A_n$ for all $n \geq 1$, then $A = \bigcup_{n\geq 1} A_n$ is integrable if and only if $\lim_{n\to\infty} \mu^*(A_n) < \infty$, in which case $\mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$.

2. If $A_{n+1} \subsetneq A_n$ for all $n \geq 1$, then $A = \bigcap_{n\geq 1} A_n$ is integrable and $\mu^*(A) = \lim_{n\to\infty} \mu^*(A_n)$.

<u>Proof</u>. 1. Suppose A is integrable. Then $\mu^*(A) < \infty$. Since $A_n \subsetneq A$ for all $n \geq 1$, $\mu^*(A_n) \leq \mu^*(A)$. Hence $\lim_{n\to\infty} \mu^*(A_n) < \infty$. Now suppose that $\lim_{n\to\infty} \mu^*(A_n) < \infty$. By the monotone convergence theorem, A is integrable and $\lim_{n\to\infty} \mu^*(A_n) = \mu^*(A)$.

2. If $A_{n+1} \subsetneq A_n$ for all $n \geq 1$, then $\{-\chi_{A_n}\}$ is increasing, and $\mu(-\chi_{A_n}) \leq 0$, and hence $-\infty < \lim_{n\to\infty} \mu(-\chi_{A_n}) \leq 0$. Thus $-\chi_A$ is integrable by the monotone convergence theorem and $\mu(-\chi_A) = \lim_{n\to\infty} \mu(-\chi_{A_n})$. But this implies that χ_A is integrable, and $\mu^*(A) = \mu(\chi_A) = \lim_{n\to\infty} \mu(\chi_{A_n}) = \lim_{n\to\infty} \mu^*(A_n)$.$\square$

Theorem 4.3. If A and B are integrable sets, then $A \cup B$, $A \cap B$, and $A - B$ are integrable.

Proof. The relations $\chi_{A \cap B} = \chi_A \cdot \chi_B$ and $\chi_{A-B} = \chi_A - \chi_A\chi_B$ imply that $A \cap B$ and $A - B$ are integrable since χ_A is integrable and $\chi_A\chi_B$ is integrable by Theorem 3.48. Note that $\chi_{A \cup B} = \chi_{A-B} + \chi_{B-A} + \chi_{A \cap B}$, and thus the function $\chi_{A \cup B}$, and hence also the set $A \cup B$, is integrable. ▯

Theorem 4.4. Relatively compact open sets and compact sets are integrable.

Proof. If G is a relatively compact open set, then $\chi_G \in LS_r^+(X)$. Now $\bar{G}$ is compact. Since X is a locally compact Hausdorff space, $\bar{G}$ has a compact neighborhood V. There exists a continuous function g mapping X into $[0, 1]$ such that $g(\bar{G}) = \{1\}$ and $g(X - V) = \{0\}$. Hence $g \in C_r^{0+}(X)$. Furthermore, $\mu^*(\chi_G) \leq \mu^*(g) = \mu(g) < \infty$ since $\chi_G \leq g$ on X. Thus χ_G is in $L(X, \mu)$ and hence G is integrable. If K is a compact set, then K has a relatively compact open neighborhood U. Then $U - K$ is relatively compact and open. Hence $U - K$ is integrable. It follows from the identity $K = U - (U - K)$ and Theorem 4.3 that K is integrable. ▯

Definition 4.5. Let A be a subset of X. Then we say that A is *μ-measurable* if and only if for every compact set K, $K \cap A$ is integrable.

We shall often omit reference to μ and state merely that A is *measurable* if no confusion can arise.

Definition 4.6. Let A be a measurable subset of X. The number $\mu^*(A)$ will be called the *measure* of A and will be denoted by $\mu(A)$.

We now investigate some properties of the set function μ as defined.

Theorem 4.7. Let A be a subset of X. Then A is integrable if and only if A is measurable and $\mu(A) < \infty$.

Proof. Suppose A is integrable. If K is any compact set, then $K \cap A$ is the intersection of two integrable sets and is integrable. Furthermore, if A is integrable, then $\mu(A) = \mu^*(\chi_A) < \infty$.

Now suppose that A is measurable and $\mu(A) < \infty$. Note that

$$\mu^*(\chi_A) = \inf \{\mu^*(\phi): \phi \in LS^+_{r^*}(X),\ \phi \geq \chi_A\} < \infty.$$

Choose $\phi \in LS^+_{r^*}(X)$ such that $\phi \geq \chi_A$ and $\mu^*(\phi) < \infty$. Now

$$\mu^*(\phi) = \sup \{\mu(f): f \in C^o_r(X),\ f \leq \phi\}.$$

For each integer $n \geq 1$, choose $f^*_n \in C^o_r(X)$ such that $f^*_n \leq \phi$ and $\mu(f^*_n) > \mu^*(\phi) - 1/n$. Now choose $f_n = \max \{f^*_1, \ldots, f^*_n\}$. Then (f_n) is an increasing sequence of functions such that $f_n \leq \phi$ and $\mu(f_n) > \mu^*(\phi) - \frac{1}{n}$. Hence $\lim \mu(f_n) = \mu^*(\phi) < \infty$. By the monotone convergence theorem it follows that $f = \lim f_n$ is integrable and that $f(x) = \phi(x)$ a.e. on X. Let $N = \{x: f(x) \neq \phi(x)\}$. Let K_n be the support of f_n. Then $\{K_n\}$ is an increasing sequence of compact sets. We shall show that $A \subseteq (\bigcup_{n\geq 1} K_n) \cup N$. Suppose $x \in A$. Then $\phi(x) \geq 1$. Either $f(x) \geq 1$ or $x \in N$. If $x \notin N$, then, for some n, $x \in K_n$. Thus $x \in (\bigcup_{n\geq 1} K_n) \cup N$. Hence $A \subseteq (\bigcup_{n\geq 1} K_n) \cup N$. Thus $A = \bigcup_{n\geq 1} (A \cap K_n) \cup (A \cap N)$. Now $A \cap N \subseteq N$ is integrable. Since A is measurable, each $A \cap K_n$ is integrable. The sequence $\{A \cap K_n\}$ is increasing and hence, by Theorem 4.2, $\bigcup_{n\geq 1} (A \cap K_n)$ is integrable since $\lim \mu(A \cap K_n) \leq \mu(A) < \infty$. Thus A is integrable by Theorem 4.3. ▯

<u>Theorem 4.8</u>. If A is a measurable set and if (A_n) is a sequence of measurable sets, then the sets $X - A$, $\bigcup_{n\geq 1} A_n$, $\bigcap_{n\geq 1} A_n$, $\underline{\lim}\, A_n$, and $\overline{\lim}\, A_n$ are measurable.

<u>Proof</u>. Let K be any compact set. Then $(X - A) \cap K = K - (A \cap K)$ which is integrable by Theorem 4.3. Hence $X - A$ is measurable. If A and B are measurable, then $A \cup B$ and $A \cap B$ are measurable since, for instance, $(A \cup B) \cap K = (A \cap K) \cup (B \cap K)$ and, by Theorem 4.3, finite unions of integrable sets are integrable. If $A_n \uparrow A$, then for any compact set K, $(A_n \cap K) \uparrow (A \cap K)$, and hence $A \cap K$ is integrable by Theorem 4.2 and the fact that $\mu^*(A \cap K) \leq \mu^*(K) < \infty$. Thus the limit of an increasing sequence of measurable sets is measurable. Combining the above two arguments we have that for any sequence (A_n) of measurable sets $\bigcup_{n\geq 1} A_n$ is measurable. A similar argument shows that $\bigcap_{n\geq 1} A_n$ is measurable. Finally, by what has just been established, $\underline{\lim}\, A_n$ and $\overline{\lim}\, A_n$ are measurable. ▯

The above result shows that the measurable subsets of X form a σ-algebra since X itself is clearly measurable.

Theorem 4.9. Open sets and closed sets are measurable.

This follows at once from the definition of measurable set.

Theorem 4.10. If (A_n) is a sequence of pairwise disjoint measurable sets and if $A = \bigcup_{n\geq 1} A_n$, then $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

Proof. Let $\bar{A}_j = \bigcup_{n=1}^{j} A_n$, for $j \geq 1$. Then

$$\mu(\bar{A}_j) = \mu^*(\bar{A}_j) = \mu^*\Big(\sum_{n=1}^{j} \chi_{A_n}\Big) \leq \sum_{n=1}^{j} \mu^*(\chi_{A_n}) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

The sequence $(\chi_{\bar{A}_j})$ is an increasing sequence, and by Fatou's theorem,

$$\mu^*(\chi_A) = \mu^*(\lim_{j\to\infty} \chi_{\bar{A}_j}) = \lim_{j\to\infty} \mu^*(\chi_{\bar{A}_j}) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

If $\mu^*(\chi_A) = \infty$, the assertion of the theorem is trivial. If $\mu^*(\chi_A) < \infty$, then A is integrable. Furthermore, $\mu^*(\chi_{A_n}) \leq \mu^*(\chi_A) < \infty$ and hence A_n is integrable for each n. Since the sets A_n are pairwise disjoint, $\chi_{\bar{A}_j} = \sum_{n=1}^{j} \chi_{A_n}$. Hence $\mu(\chi_{\bar{A}_j}) = \sum_{n=1}^{j} \mu(\chi_{A_n})$. But $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$, and by monotone convergence,

$$\mu(\chi_A) = \mu(\lim_{j\to\infty} \chi_{\bar{A}_j}) = \lim_{j\to\infty} \mu(\chi_{\bar{A}_j}) = \sum_{n=1}^{\infty} \mu(\chi_{A_n}),$$

and hence the assertion is proved. $\square$

Note that Theorem 4.10, together with properties which μ inherits from μ^*, implies that μ is a complete measure on the measurable subsets of X.

Definition 4.11. Let A be a subset of X. Then A is said to be μ-*locally null* if and only if for every compact set K, $A \cap K$ is null.

It is obvious that any μ-locally null set is measurable.

Definition 4.12. Let A be a subset of X. Then A is said to be μ-σ-*finite* or σ-*finite relative to* μ if and only if A is contained in the union of countably many open integrable sets.

Theorem 4.13. Let A be a subset of X. Then A is σ-finite relative to μ if and only if A is contained in the union of a null set and countably many compact sets.

Proof. Suppose A is σ-finite. Let $\{G_n\}$ be a collection of open integrable sets such that $A \subseteq \bigcup_{n\geq 1} G_n$. Each characteristic function χ_{G_n} is a lower semicontinuous function on X and is in $L(X, \mu)$. We know that $\mu^*(G_n) = \sup \{\mu(f): f \in C_r^{o+}(X), f \leq G_n\}$. Thus we may choose sequences of functions $\{f_{ni}\}$ such that $f_{ni}(x) \to \chi_{G_n}(x)$ a.e. on X for each n, and $\mu(f_{ni}) \to \mu^*(\chi_{G_n}) = \mu(\chi_{G_n})$ as $i \to \infty$. For each n, define $N_n = \{x \in X: \lim_{i\to\infty} f_{ni}(x) = \chi_{G_n}(x)$ is false$\}$. Then $N = \bigcup N_n$ is null by Theorem 3.32 and the fact that $\chi_N = \sup_{n\geq 1}(\chi_{N_n})$. Let K_{ni} be the support of f_{ni}. Then (K_{ni}) is a countable family of compact sets. Now suppose that $x \in A$. Then $x \in G_n$ for some n. Hence, either $f_{ni}(x) \to 1$ or $x \in N_n$. If $f_{ni}(x) \to 1$, then for some i, $x \in K_{ni}$. Thus $x \in (\bigcup K_{ni}) \cup (\bigcup N_n)$. Hence A is contained in the union of countably many compact sets and a null set.

Conversely, suppose that $A \subseteq (\bigcup K_n) \cup N$, where (K_n) is a sequence of compact sets and N is a null set. Then, since χ_{K_n} is integrable for each n and χ_N is integrable, there exists, corresponding to ε with $0 < \varepsilon < 1$, and each $n \geq 1$, a function $\phi_n \in LS^+_{r^*}(X)$ such that $\phi_n \geq \chi_{K_n}$ and $\mu^*(\phi_n) < \mu^*(\chi_{K_n}) + \varepsilon$. The set $G_n = \{x: \phi_n(x) > 1 - \varepsilon\}$ is open and $\chi_{G_n} \leq (1 - \varepsilon)^{-1} \phi_n$. In a similar manner, we obtain a function ϕ and a set G corresponding to the set N. Since

$$\mu^*(\chi_{G_n}) \leq (1 - \varepsilon)^{-1} \mu^*(\phi_n) < \infty$$

and χ_{G_n} is lower semicontinuous, G_n is integrable. It is easy to verify that G is also integrable. If $x \in A$, then either $x \in K_n$ for some n, or $x \in N$. In either case $\phi_n(x) \geq 1$ for some n or $\phi(x) \geq 1$. Hence $x \in G_n$ for some n, or $x \in G$. Thus $A \subsetneq (\bigcup_{n\geq 1} G_n) \cup G$, and A is σ-finite. ▯

Theorem 4.14. If A is an integrable subset of X, then A is σ-finite.

Proof. There exists a sequence (f_n) of functions such that $f_n \in C^o_r(X)$ for each n and $f_n \to \chi_A$ a.e. on X. Let K_n be the support of f_n. Let $N = \{x: \lim f_n(x) \neq \chi_A(x)\}$. Then N is null. Suppose $x \in A$. Then either $f_n(x) \to 1$, or $x \in N$. If $x \in N$, then $x \in (\bigcup_{n\geq 1} K_n) \cup N$. If $f_n(x) \to 1$, then for some n, $x \in K_n$. Otherwise, $f_n(x) \to 0$ for all n. Hence $x \in (\bigcup_{n\geq 1} K_n) \cup N$. It follows that A is σ-finite. ▯

Theorem 4.15. If A is σ-finite, then A is locally null if and only if A is null.

Proof. Suppose A is locally null. Since A is σ-finite, $A \subseteq (\bigcup_{n\geq 1} K_n) \cup N$, by Theorem 4.13. Hence $A = \bigcup_{n\geq 1} (A \cap K_n) \cup (N \cap A)$. Now $N \cap A$ is null since N is null, and $A \cap K_n$ is null since A is locally null. Then by Theorem 3.40, $\mu^*(A) \le \sum_{n=1}^{\infty} \mu^*(A \cap K_n) + \mu^*(N) = 0$. Hence A is null. Clearly, if A is null, then A is locally null. ▯

Definition 4.16. Let P be a property of points of X. We say that P holds μ-locally almost everywhere, abbreviated l.a.e. (μ), if and only if the set of points $N = \{x: P(x) \text{ is false}\}$ is locally null relative to μ.

Theorem 4.17. Let A be measurable. Then there exists a sequence (G_n) of open sets such that $\mu(A) = \inf \{\mu(G_n): n \ge 1\}$, $A \subseteq \bigcap_{n\ge 1} G_n$, and $A - \bigcap_{n\ge 1} G_n$ is locally null. If A is compact, the sets G_n may be taken relatively compact.

Proof. It is clear that $\mu(A) \le \inf \{\mu(G_n): n \ge 1\}$ since $A \subseteq G_n$ for each n and $\mu(A) \le \mu(G_n)$. If $\mu(A) = \infty$, then the conclusion is immediate. Suppose $\mu(A) < \infty$. Given ε, $0 < \varepsilon < 1$, there exists a function $\phi \in LS^+_{r*}(X)$ such that $\phi \ge \chi_A$ and $\mu^*(\phi) \le \mu(A) + \varepsilon$. Let $G=\{x:\phi(x)>1-\varepsilon\}$. Then G is open and $G \supseteq A$. Furthermore $\chi_G \le (1 - \varepsilon)^{-1}\phi$; hence

$$\mu(G) = \mu^*(\chi_G) \le (1 - \varepsilon)^{-1}\mu^*(\phi) \le (1 - \varepsilon)^{-1}\{\mu^*(A) + \varepsilon\}.$$

Corresponding to each $n \ge 1$, choose $\varepsilon = 1/n + 1$ and obtain G_n as above. Then

$$\mu^*(A) \le \mu(G_n) \le \frac{\mu^*(A) + \frac{1}{n+1}}{1 - \frac{1}{n+1}},$$

from which it is clear that $\mu^*(A) = \mu(A) = \inf \{\mu(G_n): n \geq 1\}$. The fact that $A - \bigcap G_n$ is locally null is easily established and the proof is omitted here. If A is compact, then A has a relatively compact open neighborhood G. The function χ_G is lower semicontinuous. In the above proof, choose ϕ_n' to be a lower semicontinuous function such that $\phi_n' \geq \chi_A$ and $\mu^*(\phi_n') \leq \mu^*(A) + \frac{1}{n+1}$. Then choose $\phi_n = \min \{\phi_n', \phi\}$. Then $\phi_n \geq \chi_A$ and $\mu^*(\phi_n) \leq \mu^*(A) + \frac{1}{n+1}$. Furthermore, the set G_n obtained above is relatively compact. $\square$

We now consider the concept of measurable function and show the relationship between measurable functions and integrable functions.

Definition 4.18. Let f be a function defined on X. Then f is said to be measurable relative to μ if and only if f is real-valued and for each real number λ, the set $\{x: f(x) > \lambda\}$ is measurable relative to μ.

Equivalent definitions of measurability for functions are obtained by substituting the sets $\{x: f(x) \geq \lambda\}$, $\{x: f(x) \leq \lambda\}$, and $\{x: f(x) < \lambda\}$ for the set $\{x: f(x) > \lambda\}$ in the above definition [6]. It is also true that sums and products of measurable functions are measurable. If f is never zero,and measurable, then $1/f$ is measurable. The infimum and supremum of a collection of measurable functions are measurable whenever they are real-valued. If $f(x) = g(x)$ a.e. on X and f is measurable, then g is measurable if g is real-valued. If (f_n) is a sequence of measurable functions, then $\underline{\lim}\, f_n$ and $\overline{\lim}\, f_n$ are measurable whenever they are real-valued. If $\lim_{n\to\infty} f_n(x) = g(x)$ a.e. on X and f_n is measurable, then g is measurable if g is real-valued [6].

Theorem 4.19. If f is a non-negative measurable function, and if $f \in F(X, \mu)$, then the set $\{x: f(x) \neq 0\}$ is σ-finite.

Proof. Let $A = \{x: f(x) \neq 0\}$. Let $\bar{f} = \min\{1, f\}$. Then $\bar{f} \in F(X, \mu)$, and $\{x: \bar{f}(x) \neq 0\} = A$. There exists a function $\bar{\phi} \in LS^+_{r*}(X)$ such that $\bar{\phi} \geq \bar{f}$ and $\mu^*(\bar{f}) < \mu^*(\bar{\phi}) < \infty$. Choose $\phi = \min\{1, \bar{\phi}\}$. Then ϕ is a real-valued lower semicontinuous function, $\phi \geq \bar{f}$ and $\mu^*(\bar{f}) \leq \mu^*(\phi) \leq \mu^*(f) < \infty$. Hence $\phi \in L(X, \mu)$ by Theorem 3.24. There exists an N-Cauchy sequence (f_n) of functions in $C_r^o(X)$ such that $f_n(x) \to \phi(x)$ a.e. on X. Let $N = \{x: \lim_{n\to\infty} f_n(x) \neq \phi(x)\}$. Then N is a null set. Let K_n be the support of f_n. If $x \in A$, then $\phi(x) > 0$. Either $f_n(x) \to \phi(x)$ or $x \in N$. If $f_n(x) \to \phi(x)$, then for some n, $x \in K_n$, since $\phi(x) > 0$. Thus $A \subseteq (\bigcup_{n\geq 1} K_n) \cup N$. Hence A is σ-finite. □

Theorem 4.20. If f is a real-valued function, then f is in $L(X, \mu)$ if and only if f is measurable and $\mu^*(|f|) < \infty$.

Proof. Suppose that $f \in L(X, \mu)$. Then it is clear that $\mu^*(|f|) < \infty$. Furthermore, we know that there exists a sequence (f_n) of functions in $C_r^o(X)$ such that $f_n \to f$ a.e. on X. Now each f_n is continuous and hence measurable. Thus $\overline{\lim} f_n$ is measurable and hence so also is f.

Now suppose that f satisfies the stated conditions. We shall first assume that the range of f is contained in $[0, 1)$ and that the support of f is contained in an integrable set A. Let m be a positive integer and suppose that $P = \{x_o, x_1, \ldots, x_n\}$ is a partition of $[0, 1]$ such that the norm $|P| < 1/m$.

Let

$$A_k = \{t \in X: x_{k-1} < f(t) \le x_k\}, \quad k = 1, 2, \ldots, n.$$

Then each A_k is measurable and contained in A. Hence each A_k is integrable by Theorem 4.7. Thus the function g_m defined by $g_m = \sum_{k=1}^{n} x_{k-1} \chi_{A_k}$ is integrable. Now let $t \in X$. If $t \notin \bigcup_{k\ge1} A_k$, then $g_m(t) = 0 = f(t) \le g_m(t) + \frac{1}{m} \chi_A(t)$. If $t \in \bigcup_{k\ge1} A_k = A$ then $t \in A_k$ for some k. Hence

$$g_m(t) = \sum_{k=1}^{n} x_{k-1} \chi_{A_k} = x_{k-1} < f(t) \le x_{k-1} + x_k - x_{k-1}$$

$$< g_m(t) + (1/m)\chi_{A_k}(t).$$

Hence $0 \le f(t) - g_m(t) \le 1/m \, \chi_A(t)$ for all $t \in X$. Thus $N(f - g_m) \le (1/m)\mu(A)$. Hence $f \in L(X, \mu)$.

We now remove the restriction that f vanishes outside an integrable set. The support of f vanishes outside a σ-finite set by the previous theorem. Hence $A \subsetneq \bigcup_{n\ge0} K_n$ where each K_n is compact, and K_o is a null set. We may arrange to have the sequence (K_n) increasing, for $n \ge 1$. Thus the sequence $f_n = f\cdot\chi_n (n \ge 1)$ is increasing and $\lim_{n\to\infty} f_n(t) = f(t)$ a.e. on X. Furthermore, if $\lim_{n\to\infty} f_n(t) \ne f(t)$, for some t in X, then $\lim_{n\to\infty} f_n(t) = 0$. Hence the limit function $\bar{f} = \lim_{n\to\infty} f_n$ is real-valued. The previous part of the proof applies to each f_n. Hence each f_n is integrable and $\mu(f_n) \le N(f) < \infty$. Hence $\lim_{n\to\infty} \mu(f_n) < \infty$. Thus $\bar{f}$ is integrable by the monotone convergence theorem. It follows that f is integrable, since f is real-valued and $\bar{f}(t) = f(t)$ a.e. on X.

It is clear that the assumption that $0 \leq f(t) < 1$ for all $t \in X$ may be replaced by the assumption that f is bounded and positive. We now drop the assumption that f is bounded. For each $n \geq 1$, the constant function $g_n(t) = n$ for all $t \in X$ is measurable. Hence the function f_n defined by $f_n(t) = \min\{f(t), n\}$ is bounded and measurable for each n and $f_n(x) \to f(x)$ everywhere on X. Again $\mu(f_n) \leq N(f) < \infty$, and hence $\lim \mu(f_n) < \infty$. Each f_n is integrable by the preceding argument. Hence f is integrable. We now drop the assumption that f is positive. If f is measurable, then so are the functions

$$f^+ = \max\{f, 0\}, \quad f^- = \max\{-f, 0\}.$$

Both f^+ and f^- are integrable, since each is positive and the above argument applies. Since $f = f^+ - f^-$, f is also integrable. $\square$

The following theorem is usually known as the *Lebesque Dominated Convergence Theorem.*

<u>Theorem 4.21</u>. Suppose that (f_n) is a sequence of real-valued measurable functions and that g is an integrable function such that $|f_n(x)| \leq g(x)$ a.e. on X. Then the function $\liminf f_n$ and $\limsup f_n$ are integrable if they are real-valued, and $\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$ $\int \limsup f_n \, d\mu \geq \limsup \int f_n \, d\mu$. If $\lim f_n$ exists, then $\int \lim f_n \, d\mu = \lim \int f_n \, d\mu$.

The proof is omitted [6].

In the entire preceding development, we have considered the extension of a linear functional μ from the space $C_r^o(X)$ to a larger space. The same procedure could be applied to some other linear functional λ, and, in general, a distinctly different result would be obtained. We now consider some of the possible relationships between two such extensions.

Defintion 4.22. Let g be a real-valued function defined on a locally compact Hausdorff space X. Then g is said to be *locally-μ-integrable* if and only if for every compact set K, the function $g\cdot\chi_K$ is integrable.

Theorem 4.23. Let g be a real-valued function defined on X. In order that the product function f_g be integrable for any $f \in C_r^o(X)$ it is sufficient that g be locally-μ-integrable.

Proof. Let $f \in C_r^o(X)$, with support K. Then since $g\cdot\chi_K$ is integrable, there exists a sequence (g_n) of functions in $C_r^o(X)$ such that $g_n(x) \to g\cdot\chi_K(x)$ a.e. on X. Then $fg_n \to fg\cdot\chi_K = fg$ a.e. on X. Now

$$|fg_n - fg_m| \leq M|g_n - g_m|$$

where M is the maximum of $|f|$ on K. Choose a positive integer P so that $N(g_n - g_m) < \varepsilon/M$ for all $n, m \geq P$. Then $N(fg_n - fg_m) < \varepsilon$. Hence we see that (fg_n) is an N-Cauchy sequence of functions in $L(X, \mu)$. It follows that fg is in $L(X, \mu)$. ▯

Definition 4.24. Let g be a positive locally-μ-integrable function on X. For each function f in $C_r^o(X)$ define $\lambda(f) = \mu(fg)$. Then λ is called the product of g by μ, and we write $\lambda = g\cdot\mu$.

Theorem 4.25. Suppose g is a positive locally-μ-integrable function. Let λ be defined as in Definition 4.24. Then λ is a positive linear functional on $C_r^o(X)$.

The proof is immediate, and is omitted.

We may apply the extension procedure developed previously to λ obtaining a space of integrable functions denoted by $L(X, \lambda)$. The integral of f, where f is in this space will be denoted by $\int f\,d\lambda$ or $\lambda(f)$. We use λ^* to denote the extension of λ corresponding to μ^*.

Theorem 4.26. Let $\lambda = g \cdot \mu$ where g is a positive locally-μ-integrable function on X. Then:

1. For every ϕ in $LS_r^+(X)$, $\lambda^*(\phi) \leq \mu^*(g\phi)$.
2. Each μ-locally null set is λ-locally null.

Proof. 1. By definition,

$$\lambda^*(\phi) = \sup \{\lambda(f)\colon f \in C_r^{o+}(X),\ f \leq \phi\}$$

$$= \sup \{\mu(gf)\colon f \leq \phi,\ f \in C_r^{o+}(X)\}.$$

Now $gf \leq g\phi$ for every f in $C_r^{o+}(X)$ such that $f \leq \phi$. Hence $\mu(gf) = \mu^*(gf) \leq \mu^*(g\phi)$. Hence

$$\lambda^*(\phi) = \sup \{\mu^*(gf)\colon f \leq \phi,\ f \in C_r^{o+}(X)\} \leq \mu^*(g\phi).$$

2. Let N be a relatively compact μ-null set. Let n be a positive integer. Then, by definition of $\mu(N)$, there exists a function

$\phi_n^* \in LS_r^+(X)$ such that

$$(1) \qquad 0 = \mu^*(N) \leq \mu^*(\phi_n^*) < \mu(N) + \frac{1}{n}$$

and $\phi_n^* \geq \chi_N$. We see that $\mu^*(\phi_n^*) \downarrow 0$. Furthermore, we may take (ϕ_n^*) to be a decreasing sequence by choosing $\phi_n^* = \inf \{\bar{\phi}_n, \phi_{n-1}^*, \ldots, \phi_1^*\}$ at each stage. Here $\bar{\phi}_n$ is chosen to satisfy (1). By Theorem 3.44, there exists a subsequence (ϕ_n') of (ϕ_n^*) such that $\phi_n' \downarrow 0$ a.e. on X. Now let V be any compact neighborhood of $\bar{N}$. There exists a continuous function f on X into [0, 1] such that $f(\bar{N}) = \{1\}$ and $f(X - V) = \{0\}$. By choosing $\phi_n = \phi_n' f$, we see that we may choose the ϕ_n in $LS_r^+(X)$ such that each ϕ_n vanishes outside a fixed compact neighborhood of N and still have $\phi_N \geq \chi_N$ for all n. Now $\lambda^*(\phi_n) \leq \mu^*(g\phi_n)$. Hence $\lambda^*(\phi_n) \to 0$, and thus $\lambda(N) = 0$. Here we have used Theorem 4.23 and the fact that the majorant $g\phi_1$ is integrable. The conclusion is now obvious. ▯

Definition 4.27. Let F be a family of subsets of X. Then F is said to be *locally countable* if and only if for each compact set K, K meets at most countably many members of F.

Theorem 4.28. Let μ be a positive linear functional on $C_r^o(X)$, and consider its extension and the measure it induces on X. There exists a locally countable family F such that

1. F is a pairwise disjoint collection of compact sets.

2. $\mu(F \cap U) > 0$ for each F in F and each open subset U of X such that $U \cap F \neq \phi$.

3. The set $X - U$ is μ-locally null.

Proof. Let $\mathcal{A}$ be the collection of all families A of compact subsets of X having properties (1) and (2). We show first that $\mathcal{A}$ is non-empty. Let $U = \{G \subsetneq X\colon G \text{ is open and } G \text{ is } \mu\text{-null}\}$. Then U is a family of open μ-null sets. For each $G \in U$, let χ_G be the characteristic function of G. Then $\chi_G \in LS_r^+(X)$. Furthermore, given any two functions χ_G and $\chi_{G'}$, we see that $\chi_{G\cup G'} \geq \chi_G$ and $\chi_{G\cup G'} \geq \chi_{G'}$. Hence the family $\{\chi_G\colon G \in U\}$ is directed upward. Now set $U = \bigcup_{G\in U} G$. Then $\mu^*(\chi_U) = \sup\{\mu^*(\chi_G)\colon G\in U\}$ by Theorem 3.13. It is clear that U is the maximal μ-null open set. Now $X - U$ is closed. If K is a compact subset of $X - U$, then the singleton family $\{K\}$ satisfies (1) and (2) above. Hence we have shown the existence of at least one such family. If $\mathfrak{F}$ is a chain in $\mathcal{A}$, then $\bigcup_{F\in\mathfrak{F}} F$ is obviously in $\mathcal{A}$. By Zorn's Lemma, there exists a maximal family F satisfying (1) and (2). We now show that F is locally countable. Let K be a relatively compact open set. Then, if $\{F_1, \ldots, F_n\}$ is any finite collection of sets in F, we have $\sum_{i=1}^{n} \mu(F_i \cap K) = \mu(\bigcup_{i=1}^{n} (F_i \cap K)) \leq \mu(K)$, since the sets F_i are pairwise disjoint. This implies that for any countable subcollection $\{F_i\}$ in F, the sum $\sum_{i=1}^{\infty} \mu(F_i \cap K)$ converges. But this then implies that $\mu(F \cap K) > 0$ for only countably many members of F. Any compact set K has a relatively compact open neighborhood containing K, and hence F is locally countable.

We now show that $N = X - \cup F$ is locally μ-null. Suppose N is not locally μ-null. Then there exists a compact set K such that $\mu(K \cap N) > 0$. Let $K' = K \cap N$. Then $K' \subsetneq N$ and $\mu(K') > 0$. Obviously $\chi_{K'}$ is locally μ-integrable. The functional $\mu_K = \chi_K \cdot \mu$ is a non-zero functional. Repeating an argument given earlier in the proof, we obtain a maximal open set U such that $\mu_K(U) = 0$. Thus the set $M = X - U$ is the smallest closed set such that $X - M$ is μ_K-null. Obviously $X - \bar{K}'$ is μ_K-null. Thus $M \subseteq \bar{K}'$. Hence M is compact, since $\bar{K}' \subseteq K$. M is non-empty since otherwise μ_K would be a zero functional. For any open set G such that $G \cap M \neq \Phi$, $\mu(G) > 0$, for otherwise, $G \subseteq U$. Now $G \cap U$ is an open subset of U, and hence $\mu_K(G \cap U) = 0$. But $\mu_K(G) = \mu_K(G \cap U) + \mu_K(M \cap G)$ implies that $\mu_K(G \cap M) = \mu_K(G)$. Thus $\mu_K(M \cap G) > 0$. Furthermore, $\mu(M \cap G) \geq \mu_K(M \cap G) > 0$. Hence $\mathcal{F}' = \mathcal{F} \cup \{M\}$ would have properties (1) and (2) thus contradicting maximality of $\mathcal{F}$. Thus N must be locally μ-null. ▯

Theorem 4.29. Let λ and μ be two distinct positive linear functionals defined on $C_r^o(X)$. Then there exists a positive function g which is locally μ-integrable such that $\lambda = g \cdot \mu$ if and only if each μ-locally null set is λ-locally null.

Proof. Theorem 4.26 shows that if $\lambda = g \cdot \mu$, the stated condition holds.

We first show that if S is a μ-measurable set, then it is λ-measurable. If S is μ-measurable, then for any compact set K, the set $A = S \cap K$ is μ-integrable. By Theorem 4.17, there exists a sequence (G_n) of relatively compact open sets such that $A \subseteq G_n$ and $\cap G_n - A$ is locally μ-null. But since each G_n is σ-finite, $\cap G_n - A$ is μ-null. Now $G = \cap G_n$ is relatively compact, and $G - A$ is μ-null. Hence $G - A$ is λ-null. Now G is λ-integrable, since G is relatively compact. Thus $G - (G - (S \cap K)) = S \cap K$ is λ-integrable. Thus S is λ-measurable.

Define $\rho = \mu + \lambda$. If a set A is μ-measurable, then it is λ-measurable and hence ρ-measurable. If a set is ρ-measurable, clearly it must be μ-measurable. Hence ρ-measurability is equivalent to μ-measurability. Suppose that A is a compact set. It has been pointed out before that $L_2(X, \rho)$ is a Banach space. By Schwarz's inequality, $L_2(X, \rho)$ is a Hilbert space. It is easily shown that if we restrict ourselves to those functions in $L_2(X, \rho)$ having supports in A, then these functions form a Hilbert space which we shall denote by $L_2(A, \rho)$. Now, for any $f \in L_2(A, \rho)$,

$$|\lambda(f)| \leq \lambda(|f|) = \lambda(|f\chi_A|) \leq \lambda(|f|^2)^{1/2} \cdot \lambda(\chi_A)^{1/2}$$

$$< \rho(A)^{1/2} \cdot \rho(|f|^2)^{1/2}$$

since $\lambda \leq \rho$. If (f_n) is an N_2-Cauchy sequence in $L^2(A, \rho)$, then

$$|\lambda(f_n) - \lambda(f_m)| \leq |\lambda(f_n - f_m)| \leq \rho(A)^{1/2}\, N_2(f_n - f_m)$$

i.e., $(\lambda(f_n))$ is a Cauchy sequence. Hence λ is continuous on $L_2(A, \rho)$. By a classical representation theorem due to F. Riesz [2], there exists a function F_A in $L_2(A, \rho)$ such that

$$\lambda(f) = < f, F_A > = \int F_A \cdot f \, d\rho, \text{ for all } f \in L_2(A, \rho).$$

Clearly $F_A \geq 0$ a.e.(ρ) for otherwise λ would not be a positive functional. The Riesz theorem also states that

$$\sup \{|\lambda(f)|: ||f|| \le 1\} = ||\lambda|| = ||F_A|| = (\int F_A^2 \, d\rho)^{1/2}.$$

But

$$\sup \{|\lambda(f)|: ||f|| \le 1\} \le \sup \{|\rho(f)|: ||f|| \le 1\} \le 1.$$

Thus $0 \le F_A \le 1$ a.e.(ρ). Define $T = \{x \in X: F_A(x) = 1\}$.

Then T is measurable, $T \subsetneq A$. Hence T is ρ-integrable. Furthermore, $\int \chi_T \, d\lambda = \int F_A \chi_T \, d\rho$. Now $F_A \chi_T = \chi_T$. Hence

$$\int \chi_T \, d\lambda = \int \chi_T \, d\rho = \int \chi_T \, d\mu + \int \chi_T \, d\lambda.$$

Thus $\int \chi_T \, d\mu = 0$. Hence T is μ-null, and so also λ-null. Hence T is ρ-null. Thus $0 \le F_A < 1$ a.e. We may arrange to have $F_A < 1$ everywhere on X by altering it on a set of ρ-measure zero. Now

$$\int f \cdot F_A \, d\rho = \int F_A \cdot f \, d\mu + \int F_A \cdot f \, d\lambda = \int f \, d\lambda.$$

Therefore

$$\int f(1 - F_A) \, d\lambda = \int F_A \cdot f \, d\mu, \; f \in L_2(A, \rho).$$

Define g_A by $(1 - F_A)g_A = F_A$. Then g_A is positive, μ-measurable, and vanishes outside A. Now for all functions f such that $g_A f$ is in $L_2(A, \rho)$,

$$(1) \qquad \int f\, d\lambda = \int \frac{1 - F_A}{1 - F_A} f\, d\lambda = \int g_A\, f\, d\mu.$$

Let $A_n = \{x \varepsilon A\colon 1 - F_A(x) > \frac{1}{n}\}$. Then $\bigcup A_n \subseteq A$. Suppose $x \varepsilon A$. Then $1 - F_A \geq 0$. Hence, for some n, $x \varepsilon A_n$. Thus $A = \bigcup A_n$. Suppose that f is bounded, μ-measurable, and vanishes outside A_n for some n. Then, $f(1 - F_A)^{-1}$ is bounded, ρ-measurable, and in $L_2(A, \rho)$. Thus $\int f\, d\lambda = \int f\, g_A\, d\mu$. Suppose now that f is positive, μ-measurable, and vanishes outside A. Define $f_n = \inf\,(f\cdot\chi_{A_n}, n)$. Then f_n is bounded, ρ-measurable, and vanishes outside A_n. Furthermore, if $x \varepsilon X - A$, it is clear that $f_n(x) = 0$, $f(x) = 0$ and $f_n(x) \uparrow f(x)$. If $x \varepsilon A$, then

$$f_n(x) = \min\,(f\cdot\chi_{A_n}, n) \leq \min\,(f\cdot\chi_{A_{n+1}}, n+1) = f_{n+1}(x).$$

Since f is finite, $f(x) < N$ for some N. Therefore $\min\,(f\cdot\chi_{A_n}(x), n) = f\cdot\chi_A(x)$, for $n \geq N$. Thus $f\cdot\chi_{A_n}(x) \to f(x)$ as $n \to \infty$. By Fatou's Theorem

$$(2) \qquad \lambda^*(f) = \mu^*(g_A\cdot f).$$

This relation holds if $f = \chi_A$. Now $\lambda(A) < \infty$ since A is compact; $g_A\cdot\chi_A = g_A$, and thus $\mu(g_A) < \infty$ since $\lambda(A) = \mu(\chi_A\cdot g_A)$. Thus g_A is integrable (μ).

We now apply Theorem 4.28, obtaining a family F of subsets of X with the properties stated there. Corresponding to each compact set F in F, obtain g_F as above. Define $g = \sup_{F \varepsilon F} \{g_F\}$. Then g is positive and μ-measurable since F is locally countable. Now let f be any positive, bounded, μ-measurable function which vanishes outside some compact set A. Then $F \cap K \neq \phi$ for at most a countable number of F's in $\mathcal{F}$, say $F_1, F_2, \ldots$. Then

$$f = \sum_{n=1}^{\infty} f\chi_{F_n}$$

and

$$gf = \sum_{n=1}^{\infty} g\chi_{F_n} f;$$

$f \cdot \chi_{F_n}$ is λ-integrable, and $f \cdot g_{F_n}$ is μ-integrable. Hence $\lambda(f \cdot \chi_{F_n}) = \mu(g_{F_n} \cdot f)$. By monotone convergence,

$$\lambda(f) = \lambda\left(\sum_{n=1}^{\infty} f \cdot \chi_{F_n}\right) = \sum_{n=1}^{\infty} \lambda(\chi_{F_n} \cdot f) = \sum_{n=1}^{\infty} \mu(g_{F_n} \cdot f) = \mu(gf).$$

Here gf is integrable, since $\sum_{n=1}^{\infty} \mu(g_{F_n} \cdot f) = \lambda(f) < \infty$. Thus g is locally integrable for μ since we may take $\chi_K = f$ for any compact set K. The last formula is valid for every f in $C_r^{o+}(X)$. Thus $\lambda = g \cdot \mu$. ▯

<u>Theorem 4.30</u>. Let λ and μ be two positive linear functionals on $C_r^o(X)$ such that $\lambda = g \cdot \mu$ for some positive μ-locally integrable function g. Let $f \in \mathcal{L}(X, \lambda)$. Then there exists a function $\bar{f} \in \mathcal{L}(X, \mu)$ such that $\int f \, d\lambda = \int \bar{f}g \, d\mu$, and $\bar{f} = f$ a.e.(λ) on the set $T = \{x: g(x) \neq 0\}$.

<u>Proof</u>. There exists an N-Cauchy(λ) sequence (f_n^*) of functions in $C_r^o(X)$ such that $\lambda(|f - f_n|) \to 0$ as $n \to \infty$. The sequence $\{g \cdot f_n^*\}$ is N-Cauchy(μ). Hence a subsequence $g \cdot f_n \to f^*$ a.e.(μ), where f^* is in $\mathcal{L}(X, \mu)$.

Define $\bar{f}$ as follows:

$$\bar{f}(x) = \begin{cases} \dfrac{f(x)}{g(x)} , & x \in T \\ 0 , & x \notin t. \end{cases}$$

Then clearly $\bar{f}$ has the stated properties. $\square$

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